UT-HEP-760-96

On the Structure of the Small Quantum Cohomology Rings of Projective Hypersurfaces

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February 1, 2008

Abstract

We give an explicit procedure which computes for degree $d \leq 3$ the correlation functions of topological sigma model (A-model) on a projective Fano hypersurface X as homogeneous polynomials of degree d in the correlation functions of degree 1 (number of lines). We extend this formalism to the case of Calabi-Yau hypersurfaces and explain how the polynomial property is preserved. Our key tool is the construction of universal recursive formulas which express the structural constants of the quantum cohomology ring of X as weighted homogeneous polynomial functions in the constants of the Fano hypersurface with the same degree and dimension one more. We propose some conjectures about the existence and the form of the recursive formulas for the structural constants of rational curves of arbitrary degree. Our recursive formulas should yield the coefficients of the hypergeometric series used in the mirror calculation. Assuming the validity of the conjectures we find the recursive laws for rational curves of degree 4 and 5. [1]

1 Introduction

In [16], we studied the Kähler sub-ring $H^*_{q,e}(M^k_N)$ in the quantum cohomology ring of a hypersuface M^k_N of degree k in CP^{N-1}, we used numerical computation based on the torus action method. We worked under the condition that $c_1(M^k_N)$ is not negative, i.e. under the ipothesis $N-k \geq 0$. The following statements summarize the content of that paper:

1.For $N \leq 9$ with $N-k \geq 2$, we computed that the main relation satisfied by the generator $\mathcal{O}_e$ of $H^*_{q,e}(M^k_N)$ has the simple form

$$(\mathcal{O}_e)^{N-1} - k^k(\mathcal{O}_e)^{k-1} \cdot q = 0 \tag{1.1}$$

2. Under the same restriction as for 1 the structural constants of $H^*_{q,e}(M^k_N)$ can be expressed as polynomial functions of a finite set of integers. These integers are the Schubert numbers of lines, they do depend from the degree of the hypersurface but not from its dimension.

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3. An explanation for (1.1) was found by looking at a toric compactification of the moduli space of maps from CP^1 to CP^{N-1}. It was said that the boundary portion of the moduli space should turn out to be irrelevant for the calculation, under the condition $N-k \geq 2$

The justification for 1. and 2. was based on numerical computations and the explanation of 3. was heuristic.

Givental [11] gave a mathematically rigorous proof of (1.1). He constructed the exact solution of the Gauss-Manin system (the deformation parameter is restricted to the Kähler deformation) associated to A-model on M_N^k by using torus action method and he showed that it satisfies the linear ODE of hypergeometric type if $N-k \geq 2$. This ODE reduces to the relation of $H^*_{q,e}(M_N^k)$ under certain limit (in his notation $\hbar \to 0$) and we have (1.1). He also treated the cases when $N-k$ is 1 or 0 and he showed that the solution above satisfies the linear ODE of hypergeometric type if

a) some multiplicative factor are added. (when $N-k=1$)

b) some multiplicative factor are added and at the same time, coordinate transformation (by mirror map) is performed. (when $N-k=0$)

With b) Givental proved the mirror symmetry conjecture, namely that topological sigma models on Calabi-Yau manifolds realized as the complete intersections of CP^{N-1} can be solved by the analysis of hypergeometric series. His proof of the symmetry seems to rely on the flat metric condition or the fact that the three point functions including identity operator do not receive quantum correction. Then the argument goes very smoothly but we can hardly see what is happening microspically in compensation for the smoothness. In this paper we try to explain (1.1) by descending induction of N. Our program is to construct recursive formulas that express the structural constants of $H^*_{q,e}(M_N^k)$ as weighted homogeneous polynomials in the structural constants of $H^*_{q,e}(M_{N+1}^k)$. Our method is based on a geometric process, which we call the specialization procedure, unfortunately it works only up to the case of cubic curves. We believe that it should be possible to find and construct universal recursive laws also for curves of higher degree. We state some ansatz on the expected structure of such formulas. If the index $N-k \geq 2$, the recursive formulas should stay the same, independently of N and k, and they must imply that the main relation if $H^*_{q,e}(M_N^k)$ is of the type given above (1.1). When the hypersurface is of Fano index is 1, then the resursion law for the Schubert numbers of lines changes, while the formulas for curves of higher degree do not. Coming to the Calabi-Yau situation, $N-k=0$, the recursion relations is modified for all degrees. The main relation of (1.1) must be changed entirely. We first computed the recursion relation for lines in case of $H^*_{q,e}(M_N^N)$ and evaluated the degree 1 part of the relation [16].The result had a structure strongly similar to the result from mirror symmetry [18] and we speculated that the above correction and the correction terms argued in [18] are closely related. On the other hand the universal recursion laws valid for the case $N-k \geq 2$ can be formally iterated by descent of dimension (while keeping the degree of the hypersurface fixed) up to the case of a Calabi-Yau. What we conjecture here, and verify in part, is that in this manner one recovers the coefficients of the hypergeometric series which appear in the mirror calculation, but we obtain them without use of the mirror conjecture. At this point the construction of the correction terms for the quantum ring can be done by the procedure that arises from the flat metric condition.

We prove the main relation (1.1) by means of the recursive formulas, modulo (q^4). It is clear from the topological selection rule that when N is large enough with respect to the degree k then the only non null quantum corrections left come from lines. We construct explicit recursion relations for $d \leq 3$ and prove (1.1) and 2. within this range by decreasing induction on N. We think that the universal recursive procedure should provide interesting information also for the case when M_N^k is a hypersurface of general type, i.e. $N < k$.

This paper is organized as follows.

In section 2, we recall first the main properties of the structure of the quantum Kähler algebra $H^*_{q,e}(M_N^k)$ and then we study the quantum product with primitive cohomology classes for the Fano case, $k < N$.

In section 3, we introduce the specialization calculation and derive the recursion relations for rational curves of degree at most 3 under the assumption that the hypersurface is a Fano manifold.

In section 4, we extend the specialization procedure to Calabi-Yau hypersurfaces and determine how the recursion relation should be modified (up to degree 2). Using these results, we evaluate the main relation of $H^*_{q,e}(M_N^N)$ and compare it with the result from mirror symmetry. We will also show that our findings can be organized in a compact form by means of the hypergeometric series used in mirror calculation. This is our reason for a conjecture which says how to modify the recursion relation in the case of hypersurfaces of Calabi-Yau type.

In section 5, we present a set of conjectures, which should provide a guiding rule in the explicit construction of the recursive formulas for rational curves of higher degree. Assuming the conjectures we explicitly construct the recursive formulas for degree 4 and 5.

2 Quantum cohomology of Fano hypersurfaces

2.1 The quantum Kähler Sub-Ring $H^*_{q,e}(M_N^k)$

Let M_N^k be the hypersurface of degree k in CP^{N-1}. By the Lefschetz theorem the cohomology ring $H^*(M_N^k)$ splits into two parts. One of them is the Kähler sub-ring generated by the Kähler form e induced from the hyperplane section H of CP^{N-1}, and the other is the primitive part, which is a subspace of the middle dimension cohomology $H^{N-2}(M_N^k)$. We first consider the quantum Kähler sub-ring $H^*_{q,e}(M_N^k)$, it is generated additively by $\mathcal{O}_{e^\alpha}\quad(\alpha=0,1,2,\cdots N-2)$, where $\mathcal{O}_{e^\alpha}$ represents the BRST- closed operator induced from $e^\alpha\in H^*(M_N^k)$. The multiplication rules of $H^*_{q,e}(M_N^k)$ are determined by means of the flat metric and the three point correlation functions (or Gromov-Witten invariants):

$$\begin{aligned}
\eta^{N,k}_{\alpha\beta} &:= \langle\mathcal{O}_{e^0}\mathcal{O}_{e^\alpha}\mathcal{O}_{e^\beta}\rangle_{M_N^k}=\int_{M_N^k}e^\alpha\wedge e^\beta=k\cdot\delta_{\alpha+\beta,N-2} && (2.2)\\
\eta^{\alpha\beta}_{N,k}\eta^{N,k}_{\beta\gamma} &= \delta^\alpha_\gamma,\quad \eta^{\alpha\beta}_{N,k}=\frac{1}{k}\cdot\delta_{\alpha+\beta,N-2} && (2.3)\\
C^{N,k}_{\alpha,\beta,\gamma} &= \langle\mathcal{O}_{e^\alpha}\mathcal{O}_{e^\beta}\mathcal{O}_{e^\gamma}\rangle_{M_N^k}=\sum_{d=0}^{\infty}q^d\int_{\mathcal{M}_{0,d,3}^{M_N^k}}\phi_1^*(e^\alpha)\wedge\phi_2^*(e^\beta)\wedge\phi_3^*(e^\gamma) && (2.4)\\
\phi_i:\mathcal{M}_{d,0,3}^{M_N^k} &\mapsto M_N^k,\quad (\phi_i(\{z_1,z_2,z_3,f\}/\sim)=f(z_i))\\
q &:= e^t.
\end{aligned}$$

The rules of quantum multiplication are

$$\mathcal{O}_{e^\alpha}\cdot\mathcal{O}_{e^\beta}=C^{N,k}_{\alpha,\beta,\gamma}\eta^{\gamma\delta}\mathcal{O}_{e^\delta}=\frac{1}{k}C^{N,k}_{\alpha,\beta,\gamma}\mathcal{O}_{e^{N-2-\gamma}}:=\sum_{d=0}^{\infty}q^d\frac{1}{k}C^{N,k,d}_{\alpha,\beta,\gamma}\mathcal{O}_{e^{N-2-\gamma}}. \tag{2.5}$$

We recall that $\mathcal{M}_{0,d,3}^{M_N^k}$ represents the moduli space of rational curves of degree d with three punctures in M_N^k. One should note that $\mathcal{O}_e$ is a multiplicative generator of $H^*_{q,e}(M_N^k)$, and therefore it is enough to determine the multiplication rule between $\mathcal{O}_e$ and $\mathcal{O}_{e^\alpha}$. The topological selection rule yields that $C^{N,k}_{1\alpha\beta}$ is non-zero only if $1+\alpha+\beta=N-2+(N-k)d$, hence it is:

$$\mathcal{O}_e\cdot\mathcal{O}_{e^\alpha}=\sum_{d=0}^{\infty}q^d\frac{1}{k}C^{N,k,d}_{1,\alpha,N-3-\alpha+(N-k)d}\mathcal{O}_{e^{\alpha+1-(N-k)d}}. \tag{2.6}$$

For conventional reason, we rewrite (2.6) as follows:

$$\begin{aligned}
\mathcal{O}_e\cdot\mathcal{O}_{e^{N-2-m}} &= \mathcal{O}_{e^{N-1-m}}+\sum_{d=1}^{\infty}q^dL^{N,k,d}_m\mathcal{O}_{e^{N-1-m-(N-k)d}}\\
L^{N,k,d}_m &:= \frac{1}{k}C^{N,k,d}_{1,N-2-m,m-1+(N-k)d} && (2.7)
\end{aligned}$$

Here we have used the fact that q^0 part of $H^*_{q,e}(M^k_N)$ coincides with the classical cohomology ring. The integers $L_m^{N,k,d}$ are the structural constants of the quantum ring. One should think of $kL_m^{N,k,d}$ as the number of rational curves of degree d on M^k_N which meet a linear section of dimension m and a second linear section of the right ($= m+(N-k)d-1$) codimension. We shall see that $L_m^{N,k,1}$ are independent of N if $N \geq k+2$, and therefore we write them simply as $L_m^{k,1}$; we refer to them as the Schubert number of lines. Note that $L_m^{k,1} = L_{k-m-1}^{k,1}$.

The preceding vanishing conditions translate into

$$
\begin{aligned}
L_m^{N,k,d} \neq 0 \quad &\Longrightarrow \quad 0 \leq m \leq (N-1)-(N-k)d \quad (N-k \geq 2) \\
&\Longrightarrow \quad 1 \leq m \leq (N-3) \quad (N-k=1, d=1) \\
&\Longrightarrow \quad 0 \leq m \leq (N-1)-(N-k)d \quad (N-k=1, d \geq 2) \\
&\Longrightarrow \quad 2 \leq m \leq (N-3) \quad (N-k=0)
\end{aligned}
\tag{2.8}
$$

We remark explicitly that if the dimension N is large with respect to k ($N \geq 2k$) then the only non trivial quantum correction left is due to curves of degree 1.

As we said above $\mathcal{O}_e$ is a multiplicative generator of the ring, and then there are coefficients γ which give the representations:

$$
\mathcal{O}_{e^{N-1-m}} = (\mathcal{O}_e)^{N-1-m} - \sum_{d=1}^{\infty} q^d \gamma_m^{N,k,d} (\mathcal{O}_e)^{N-1-m-(N-k)d} \tag{2.9}
$$

When we set $m=0$ we obtain the main relation of $H^*_{q,e}(M^k_N)$. One has

$$
\begin{aligned}
&\mathcal{O}_e \cdot ((\mathcal{O}_e)^{N-2-m} - \sum_{d=1}^{\infty} q^d \gamma_{m+1}^{N,k,d} (\mathcal{O}_e)^{N-2-m-(N-k)d}) \\
= \quad &(\mathcal{O}_e)^{N-1-m} - \sum_{d=1}^{\infty} q^d \gamma_m^{N,k,d} (\mathcal{O}_e)^{N-1-m-(N-k)d} \\
+ \quad &\sum_{d=1}^{\infty} L_m^{N,k,d} q^d ((\mathcal{O}_e)^{N-1-m-(N-k)d} - \sum_{d'=1}^{\infty} q^{d'} \gamma_{m+(N-k)d}^{N,k,d'} (\mathcal{O}_e)^{N-1-m-(N-k)(d+d')})
\end{aligned}
\tag{2.10}
$$

and therefore it is:

$$
\gamma_m^{N,k,d} - \gamma_{m+1}^{N,k,d} = L_m^{N,k,d} - \sum_{d'=1}^{d-1} L_m^{N,k,d-d'} \gamma_{m+(N-k)(d-d')}^{N,k,d'}. \tag{2.11}
$$

This yields:

$$
\gamma_m^{N,k,d} = \sum_{l=1}^{d} \sum_{\sum_{i=1}^{l} d_i = d} (-1)^{l-1} \sum_{j_l=m}^{N-1-(N-k)d} \cdots \sum_{j_2=m}^{j_3} \sum_{j_1=m}^{j_2} (\prod_{i=1}^{l} L^{N,k,d_i}_{j_i + (\sum_{n=1}^{i-1} d_n)(N-k)}) \tag{2.12}
$$

The fact that the main relation of $H^*_{q,e}(M^k_N)$ is of the form of $(\mathcal{O}_e)^{N-1} - k^k (\mathcal{O}_e)^{k-1} \cdot q = 0$ [16, 11] is equivalent to

$$
\begin{aligned}
\gamma_0^{N,k,1} \quad &= \quad \sum_{j=1}^{N-1} L_j^{N,k,1} = k^k \\
\gamma_0^{N,k,d} \quad &= \quad \sum_{l=1}^{d} \sum_{\sum_{i=1}^{l} d_i = d} (-1)^{l-1} \sum_{j_l=0}^{N-1-(N-k)d} \cdots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} (\prod_{i=1}^{l} L^{N,k,d_i}_{j_i + (\sum_{n=1}^{i-1} d_n)(N-k)}) = 0 \\
&\qquad (d \geq 2).
\end{aligned}
\tag{2.13}
$$

2.2 The role of primitive cohomology

In this subsection we consider the general structure of the quantum cohomology ring of a Fano hypersurface V of degree k in $\mathbf{P}^{n+1}$ ($n \geq 3$) including the primitive part.

It is $H_2(V, \mathbf{Z}) = \mathbf{Z}q$, where kq is the class of a plane section, and $H^2(V, \mathbf{Z})$ is spanned by the class $x(:= e)$ of the hyperplane section H. The ring $H^*(V, \mathbf{Q})$ is generated by x and by the primitive cohomology $H^n(V, \mathbf{Q})_0$, with the relations $x^{n+1} = 0$, $x \cup a_1 = 0$, $a_1 \cup a_2 = k^{-1} \int_V (a_1 \wedge a_2) x^n$ for a_1, a_2 primitive classes. We shall denote $(\,|\,)_V$ the intersection form, hence $(a|b)_V = \int_V a \wedge b$. For $0 \leq i \leq n$, $x_i(:= e^i)$ is the class of the linear section of V of codimension i, so that $x = x_1$. The vectors x_i span the invariant part R of $H^*(V, \mathbf{Q})$, this is the orthogonal complement of $H^n(V, \mathbf{Q})_0$. We recall that the Fano index of V is $h = h(V) = n + 2 - k$. Denote $\mathbf{Z}\{H_2(V, \mathbf{Z})\}$ the graded homogeneous ring of formal series $\sum n_d q^d$ with integer coefficients. One introduces a ring structure on $H^*(V, \mathbf{Z}\{H_2(V, \mathbf{Z})\})$ by the rule that for homogeneous α^*, β^* in $H^*(V, \mathbf{Z})$ the quantum multiplication product is $\alpha^* \cdot \beta^* = \sum_l (\alpha^*, \beta^*)_d q^d$, where $(\alpha^*, \beta^*)_0$ is the ordinary cohomology product, and $(\alpha^*, \beta^*)_d$ is a class of degree $deg(\alpha^*) + deg(\beta^*) - 2hd$ defined by the condition $((\alpha^*, \beta^*)_d|\gamma) = [\alpha^*, \beta^*, \gamma; d; V](= \langle \mathcal{O}_{\alpha^*} \mathcal{O}_{\beta^*} \mathcal{O}_\gamma \rangle_{V,d,gravity})$. This last term is the GW invariant, which can be informally defined as the number of rational curves of degree d on V meeting representative submanifold A,B,G in general position. We shall use the associativity and the grading properties of $\cdot$, whose rigorous and highly non trivial construction is due to Ruan and Tian [27]. We recall some facts from [28]. Tian observed that the GW classes $[\alpha_1, \ldots, \alpha_l; d; V]$ are invariant under monodromy action, this is a direct corollary of the main result in [RT], and he applied this explicitly to cases like hypersurfaces by using the Picard-Lefschetz theorem.

Proposition 1 Tian *If $m - l$ is odd and a_s are primitive classes then*

$$[x_{i_1}, \ldots, x_{i_l}, a_{l+1}, \ldots, a_m; d; V] = 0.$$

Proof The statement holds when n is odd for trivial reasons, indeed by definition $[x_{i_1}, \ldots, x_{i_l}, a_{l+1}, \ldots, a_m; j; V] = 0$ if $2(\sum i_j) + (m - l)n \neq 2n + 2hd + 2(m + l - 3)$. Coming to the case when the hypersurface V is even dimensional, we recall that the monodromy group M is generated by reflections defined by the vanishing cycles. The case of even dimensional quadrics is readily checked, since the vanishing cohomology has rank one in this case. On the other hand if $n > 3$ and $k > 3$, by the same argument explained in p.384 of [26], a lemma of Deligne yields that the Zariski closure $\bar{M}$ is in fact the full group of isometries of $H^*(V, \mathbf{C})_0$. Thus the GW invariant above defines a symmetric multilinear form with an odd number of entries, invariant under the orthogonal group, it is clear that such a form vanishes.

If $h \geq 2$ Tian's result yields $x \cdot a = 0$, for $a \in H^n(V, \mathbf{Q})_0$. Instead we have

Proposition 2 *If $h = 1$ then $x \cdot a = k!aq$.*

Proof The statement is equivalent to $[x, a_1, a_2; 1] = -k!(a_1|a_2)_V$, here a_i are primitive classes and $[x, a_1, a_2; 1]$ is the GW number of the lines which meet them. Our proof of this equality is based on a remark of Beauville, [1] 4. Application II. In this direction we also need to prove the formula below, which is a generalization of a result of Tyurin, [29], [3] and [22]. Let W be a general hypersurface whose generic hyperplane section is V. Then the Fano variety $F(W)$ of lines on W is non singular irreducible of dimension k and there are $k!$ lines on W which meet a general point, [22] . The variety $F(V)$ is a non singular subvariety of codimension 2 in $F(W)$. The natural P^1 bundle $p : L \to F(W)$ surjects $\lambda : L \to W$ with degree $k!$. We denote $\gamma : BF \to V$ the restriction of λ to V, γ has degree $k!$. Then $\beta : BF \to F(W)$ is the blow up along $F(V)$ and the projection of the exceptional divisor $\pi : E \to F(V)$ is the restriction of p. We denote here $i : E \to BF$ and $j : V \to W$ the natural inclusions. The cohomology of a blow up decomposes as a direct sum, in our case $H^*(BF, \mathbf{Q}) = i_*\pi^*(H^{*-2}(F(V), \mathbf{Q})) \oplus \beta^*(H^*(F(W), \mathbf{Q}))$. Now $\gamma_* H^*(BF, \mathbf{Q}) \to H^*(V, \mathbf{Q})$ is a surjection, because $\gamma : BF \to V$ is. It is known that the primitive cohomology is contained in the image $(\gamma i)_*\pi^*(H^{*-2}(F(V), \mathbf{Q}))$, [22]. We need the stronger result that given a primitive class

a there is a class α with $\gamma^*(a) = i_*\pi^*\alpha$. To prove this statement we first note that it is equivalent to $\beta_*\gamma^*(a) = 0$, and then we consider a cycle A which represents a and which is in general position with respect to the locus covered by the lines on V. We have $\beta_*\gamma^*(A) = \beta_*(\lambda^*(j_*(A)) \cap BF)$, and then $j_*(A) = 0$ in $H^{n+2}(W, \mathbf{Q}))$ because primitive classes are annihilated by j_*. Fix next a_1 and a_2 primitive classes so that $\gamma^* a_1 = i_*\pi^*\alpha_1$, $\gamma^* a_2 = i_*\pi^*\alpha_2$. One has equality of degrees of intersection $(\gamma^* a_1|\gamma^* a_2)_{BF} = k!(a_1|a_2)_V$, because the degree of γ is $k!$. On the other hand the excess intersection formula of [9] yields $(i_*\pi^*\alpha_1|i_*\pi^*\alpha_2)_{BF} = -(\pi^*\alpha_1|\pi^*\alpha_2 \cdot \zeta)_E = -(\alpha_1|\alpha_2)_{F(V)}$, here ζ denotes the tautological class of E as a P^1 bundle, and ζ is known to be the opposite of the class of the normal bundle of E in BF. Thus

$$k!(a_1|a_2)_V = -(\alpha_1|\alpha_2)_{F(V)}.$$

Now it is geometrically clear, and this is the idea from [1], that

$$[x, a_1, a_2; 1] = (\pi_* i^* \gamma^* a_1|\pi_* i^* \gamma^* a_2)_{F(V)} = (\alpha_1|\alpha_2)_{F(V)}.$$

Tian's vanishing implies also that the quantum product of the hyperplane class with a linear section is of type

$$x \cdot x_{s-1} = x_s + \sum_{i \geq 1} a_{d,s} x_{s-dh} q^d,$$

where $a_{d,s} = k^{-1}[x_{s-1}, x_{n+dh-s}, x; d]$ are the structural constants.

We set $w := x + k!q$, if $h = 1$, and otherwise $w := x$, and we write w^s the $s - th$ power of w with respect to the quantum product. Then w satisfies a unique minimal monic equation $F = 0$, of degree $(n + 1)$, the equation which is found by setting $s = n + 1$ in the displayed formula. This is of the form $F := w^{n+1} + \sum_{d=1}^{[(n+1)/h]} c_d w^{n+1-dh} q^d$. For primitive classes a and b we have $a \cdot b = k^{-1}(a|b)_V (w^n + \sum_{d=1}^{[n/h]} b_d w^{n-dh} q^d)$. Following Tian we note that associativity yields $0 = (w \cdot a) \cdot b = k^{-1}(a|b)_V (w^{n+1} + \sum_{d=1}^{[n/h]} b_d w^{n+1-dh} q^d)$, and thus $c_d = b_d$ for $1 \leq d \leq n$, $c_{n+1} = 0$. Beauville in [1] studied the structure of the quantum ring of Fano hypersurfaces of degree small with respect to the dimension. Beauville's result deals with the case $n \geq 2k - 3$, in this case only the coefficient $c_1 \neq 0$. Now $-c_1$ is the sum of the Schubert numbers of lines on V, and it turns out that $-c_1 = k^k$ hence:

Theorem 1 *The quantum cohomology of V over the rational numbers is generated by w and $H^n(V, \mathbf{Q})_0$ with relations (i) $w^{n+1} = k^k w^{k-1} q$, (ii) $w \cdot a = 0$, (iii) $a \cdot b = k^{-1}(a|b)_V (w^n - k^k w^{k-2} q)$.*

This theorem holds in fact always, the hardest part (i) is a deep theorem of Givental [11], while (ii) and (iii) follow from the same arguments used before.

3 Recursion relations for the structure constants of Fano hypersurfaces.

This section is devoted to the proof of the following recursion laws and of some related results:

Theorem 2 *Consider a hypersurface M_N^k in CP^{N-1} of degree k, if the 1st Chern class $N - k \geq 2$ then the basic structure constants satisfy the following recursion relations:*

$$L_m^{N,k,1} = L_m^{N+1,k,1} := L_m^k \tag{3.14}$$

$$L_m^{N,k,2} = \frac{1}{2}(L_{m-1}^{N+1,k,2} + L_m^{N+1,k,2} + 2L_m^{N+1,k,1} \cdot L_{m+(N-k)}^{N+1,k,1}) \tag{3.15}$$

$$L_m^{N,k,3} = \frac{1}{18}(4L_{m-2}^{N+1,k,3} + 10L_{m-1}^{N+1,k,3} + 4L_m^{N+1,k,3}$$

$$
\begin{aligned}
&+12L_{m-1}^{N+1,k,2}\cdot L_{m+2(N-k)}^{N+1,k,1}+9L_{m}^{N+1,k,2}\cdot L_{m+2(N-k)}^{N+1,k,1}\\
&+6L_{m}^{N+1,k,2}\cdot L_{m+1+2(N-k)}^{N+1,k,1}\\
&+6L_{m-1}^{N+1,k,1}\cdot L_{m-1+(N-k)}^{N+1,k,2}+9L_{m}^{N+1,k,1}\cdot L_{m-1+(N-k)}^{N+1,k,2}\\
&+12L_{m}^{N+1,k,1}\cdot L_{m+(N-k)}^{N+1,k,2}\\
&+18L_{m}^{N+1,k,1}\cdot L_{m+(N-k)}^{N+1,k,1}\cdot L_{m+2(N-k)}^{N+1,k,1}) \qquad (3.16)
\end{aligned}
$$

Our arguments are heuristic. We embed $X := M_N^k$ as the linear section of a general hypersurface $Y := M_{N+1}^k$ in CP^N so that

$$
M_N^k = M_{N+1}^k \cap H \qquad (3.17)
$$

where the hyperplane H is identified with CP^{N-1}. Next we introduce the notation

$$
\langle \mathcal{O}_{e^{a_1}} \mathcal{O}_{e^{a_2}} \cdots \mathcal{O}_{e^{a_m}} \rangle_{M_N^k,d,gravity} = [A_{a_1}^N, A_{a_2}^N, \cdots, A_{a_m}^N; d, N, k]. \qquad (3.18)
$$

Here the spaces $A_{a_i}^N$ are linear subspaces of codimension a_i in CP^{N-1} and in general position, so that

$$
PD_{M_N^k}(e^{a_i}) = A_{a_i}^N \cap M_N^k. \qquad (3.19)
$$

We define below the "special position" correlation functions.

$$
G[A_{a_1}^{N+1} \cap H, A_{a_2}^{N+1} \cap H, \cdots, A_{a_m}^{N+1} \cap H; d, N+1, k] \qquad (3.20)
$$

Clearly $A_{a_i}^{N+1} \cap H$'s is a linear subspace in CP^N of codimension $a_i + 1$ which lies in $CP^{N-1} = H$. The special position correlation function should count the number of rational curves of degree d on Y with m labeled points on them which belong to the corresponding linear spaces and which have the further property that points with different labels stay distinct. By taking the linear spaces in general position in CP^{N-1} we may assume that $[A_{a_1}^{N+1} \cap H, A_{a_2}^{N+1} \cap H, \cdots, A_{a_m}^{N+1} \cap H; d, N, k]$ has no contribution from reducible curves on X. Now an irreducible curve of degree d which cuts H in $d+1$ points lies on it and then

$$
\begin{aligned}
& [A_{a_1}^N, A_{a_2}^N, \cdots, A_{a_{d+1}}^N; d, N, k] + R \\
= \; & G[A_{a_1}^{N+1} \cap H, A_{a_2}^{N+1} \cap H, \cdots, A_{a_{d+1}}^{N+1} \cap H; d, N+1, k] \qquad (3.21)
\end{aligned}
$$

where R measures the contributions due to the connected reducible curves on Y which satisfy the conditions. In the cases the we consider R does not occur for lines and conics and it is a finite set for the case of cubic curves, as we compute below. For curves of degree 4 or more the family of reducible curves supporting R may be of positive dimension and we are not able to determine the contribution due to them. For this reason we shall restrict to the case of curves of degree d at most equal to 3.

The following lemma, the specialization formula, gives a procedure for computing the degree of $G[A_{a_1}^{N+1} \cap H, A_{a_2}^{N+1} \cap H, \cdots, A_{a_{d+1}}^{N+1} \cap H; d, N+1, k]$, because by definition $G[A_{a_1+1}^{N+1}, A_{a_2+1}^{N+1}, \cdots, A_{a_m+1}^{N+1}; d, N+1, k] = [A_{a_1+1}^{N+1}, A_{a_2+1}^{N+1}, \cdots, A_{a_m+1}^{N+1}; d, N+1, k]$. By moving $A_{a_{s+1}+1}^{N+1}$ into $A_{a_{s+1}}^{N+1} \cap H$ one has

Lemma 1

$$
\begin{aligned}
& G[A_{a_1}^{N+1} \cap H, \cdots, A_{a_s}^{N+1} \cap H, A_{a_{s+1}+1}^{N+1}, A_{a_{s+2}+1}^{N+1}, \cdots, A_{a_{s+t}+1}^{N+1}; d, N+1, k] \\
= \; & G[A_{a_1}^{N+1} \cap H, \cdots, A_{a_s}^{N+1} \cap H, A_{a_{s+1}}^{N+1} \cap H, A_{a_{s+2}+1}^{N+1}, \cdots, A_{a_{s+t}+1}^{N+1}; d, N+1, k] + \\
& \sum_{j=1}^{s} G[A_{a_1}^{N+1} \cap H, \cdots, A_{a_{j-1}}^{N+1} \cap H, A_{a_j + a_{s+1}}^{N+1} \cap H, A_{a_{j+1}}^{N+1} \cap H, \cdots, A_{a_s}^{N+1} \cap H, \\
& A_{a_{s+2}+1}^{N+1}, \cdots, A_{a_{s+t}+1}^{N+1}; d, N+1, k]
\end{aligned}
\qquad (3.22)
$$

Here we explain the definition of the special position G-W invariants.

Given a projective variety Z Kontsevich [K] has constructed the coarse moduli space $\bar{M} := \bar{M}(Z, m, \beta)$ of stable maps of homological class β to Z. $\bar{M}$ is the set of equivalence classes of data $[C, p_1, \ldots, p_m, \mu]$ where $\mu : C \to Z$ is the 'stable' map, C is a varying,projective, connected, nodal curve of arithmetic genus 0, and $p_1, \ldots, p_m$ are distinct, labeled nonsingular points on C. We refer to [FP] for a detailed discussion of this construction. The canonical evaluation maps $\rho_i : \bar{M} \to Z$ are defined by $\rho_i([C, p_1, \ldots, p_m, \mu]) = \mu(p_i)$. The interior $M(Z, m, \beta)$ is the locus in $\bar{M}(Z, m, \beta)$ corresponding to nonsingular irreducible domain curves. We write $\bar{M}(Z, A, \beta)$ when the index set of labels is a set A instead of $[n] = \{1, \ldots\ , n\}$. There are forgetful maps $\phi_B : \bar{M}(Z, A, \beta) \to \bar{M}(Z, A - B, \beta)$, defined when B is a subset of A. Let now Z be a general non singular Fano hypersurface of dimension $n \geq 3$ and index $h(Z) = n + 2 - deg(Z)$. We consider the case when β is the class of a curve of degree d and we assume that $\bar{M}(Z, m, d)$ has the expected dimension $dimZ + dh(Z) + m - 3$ and similarly for the boundary components. We recall that such components are associated with the choice of a partition $A \cup B$ of the set $[m] := \{1, \ldots, m\}$ and of the choice of d_1 and d_2 with $d = d_1 + d_2$. The boundary component $\bar{D}(A, B; d_1, d_2)$ is defined as the locus of moduli points corresponding to reducible domain curve $C = C_1 \cup C_2$, where $\mu_*(C_i)$ has degree d_i. Here the curve C is obtained by gluing at $\bullet$ the curve C_1 which has on it points marked by the elements in A and a further point, labeled by $\bullet$. and C_2, which has on it points marked by the elements in B and a further point, also labeled by $\bullet$. There is an identification $\bar{D}(A, B; d_1, d_2) = \bar{M}(Z, A \cup \{\bullet\}, d_1) \times_Z \bar{M}(Z, B \cup \{\bullet\}, d_2)$.

In what follows we take $n + 2 = N$ and define T_i, $i = 1, ..., m$ to be linear spaces of codimension $t_i \geq 1$ in P^{n+2} and in general position there. As before Y is a Fano hypersurface of degree k. We assume that $\sum t_i$ is the expected dimension $dimY + dh(Y) + m - 3$ of $\bar{M}(Y, m, d)$. We define $[t_1, ..., t_m; d, Y]$ to be the degree of the zero cycle $[T_1, ..., T_m; d, Y]$, which we define to be the intersection product of the cycles $\rho_i^{-1}(T_i)$. Here we assume that those cycles intersect transversally in a finite number of points, each one which is associated with an irreducible source curve and with the property that the corresponding map sends different labeled points to different images. By definition $[t_1, ..., t_m; d, Y]$ is one of the GW invariants of Y, it is called basic if $m = 3$ and at least one of the t_i is 1. The GW invariants on X are defined in a similar way, by means of linear spaces $S_i \subset P^{n+1}$. We shall use the convention that S_i and T_i are spaces of the same dimension, so that S_i is obtained by moving T_i into P^{n+1}.

Given linear spaces as above we write $G[S_1, \ldots, S_s, T_{s+1}, \ldots, T_{s+t}; d; Y]$ to represent the open cycle in $\bar{M}(Y, s+t, d)$ which can be informally described as the set of rational curves of degree δ on Y with $s+t$ marked points such that the images of the labeled points p_j belong to the space with the same label, and such that for $j \leq s$ and $i \leq s$ if p_j and p_i have the same image point in $S_j \bigcap S_i$ then this point is a double point for the image curve. We shall use the notation that s_i is the codimension of S_i in P^{n+1} so that $s_i = t_i + 1$, because of our convention. The codimension of the preceding cycle is $\sum_j (s_j + 1) + \sum_j t_j$. Our aim is to compute the degree of $G[S_1, \ldots, S_s, T_{s+1}, \ldots, T_{s+t}; d; Y]$ when its expected dimension is 0. By abuse of notations we shall often use the same notation to represent both a cycle of dimension 0 and the degree of the said cycle. We define $\bar{M}^0(s)$ to be the complement in $\bar{M} := \bar{M}(Y, s + t, d)$ of the union of the components of type $\bar{D}(A, B; d_1, d_2)$ with $d_2 = 0$ and with at least two elements of B which are $\leq s$. Thus we have $\bar{M}^0(s+1) \subset \bar{M}^0(s) \subset \bar{M}$. The evaluation ρ_i restricts to $\rho(s)_i^0$ on $\bar{M}^0(s)$. Our definition is that $G[S_1, ..., S_s, T_{s+1}, ..., T_{s+t}; d; Y]$ is the intersection product of the cycles $(\rho(s)_j^o)^{-1}(S_j)$,$j = 1, ...s$, $(\rho(s)_l^o)^{-1}(T_l)$, $l = s+1, ..., s+t$. If the codimensions s_j and t_l are fixed the set of lists $(S_1, ..., S_s, T_{s+1}, ..., T_{s+t})$ is parameterized by a product of Grassmann manifolds, hence it is an irreducible variety and then there is an open dense subset of it where the degree of $G[S_1, ..., S_s, T_{s+1}, ..., T_{s+t}]$ is maximum. We shall assume that our lists come from this subset.

We start by noting that $[S_1, ..., S_{d+1}; d; X]$ and $G[S_1, ..., S_{d+1}; d; Y]$ both have the same expected dimension, which we take to be 0.

Proposition 3 *If the given cycles are zero dimensional then*

$G[S_1, ..., S_{d+1}; d; Y] = [S_1, ..., S_{d+1}; d; X] + R$,

where R is supported on the boundary locus of $\bar{M}(Y, m, d)$ and more precisely on the locus corresponding to reducible domain curves with reducible image.

In order to compute the degree of $[S_1, ..., S_{d+1}; d; X]$ we need to verify that the dimension of the preceding cycles is in fact 0 and then to compute their degrees. Now we have by assumption that $[S_1, ..., S_{d+1}; d; X]$ has the correct dimension 0, so the dimension of $G[S_1, ..., S_{d+1}; d; Y]$ can fail to be also $= 0$ only if R fails. Of course the dimension of R can be detected by looking at decomposable curves on Y, that is to the behavior of rational curves of degree strictly less than d.

As we have said above the degree of $G[S_1, ..., S_{d+1}; d; Y]$ is determined by a reduction procedure, which is performed by moving linear spaces T_i which are in general position in P^{n+2} to spaces S_i of the same dimension, which are contained in the hyperplane P^{n+1} and in general position there. Our main tool is next proposition, we have only heuristic arguments to support it

Proposition 4 *Provided that the dimensions of the cycles below are 0 as it is expected then* $degree G[S_1, ..., S_s, T_{s+1}, ..., T_{s+t}; d; Y] =$

$degree G[S_1, ..., S_s, S_{s+1}, T_{s+2}, ..., T_{s+t}; d; Y]\ +$

$degree \sum \psi_i^{-1} G[S_1, \ldots, S_{i-1}, S_{i,s+1}, S_{i+1}, \ldots, S_s, T_{s+2}, \ldots, T_{s+t}; d; Y]$,

where $S_{i,s+1} := S_i \cap S_{s+1}$ and where ψ_i is the isomorphism $\bar{D}([s+t] - \{i, s+1\}, \{i, s+1\}; d, 0) \to \bar{M}(Y, ([s+t] - \{i, s+1\}) \cup \{\bullet\}, d)$.

The specialization lemma is just a restatement of this proposition.

The following procedure gives the recursive formulas :

1. By iterative application of the specialization formula we write $G[A_a^{N+1} \cap H, A_b^{N+1} \cap H, A_1^{N+1,1} \cap H, \cdots, A_1^{N+1,d-1} \cap H; d, N+1, k]$ in terms of the standard correlation functions of $Y = M_{N+1}^k$.

2. We decompose the standard correlation functions found in step 1 as polynomials in the basic G-W invariants of Y, by which we mean the functions $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_e \rangle_{d, M_{N+1}^k, gravity}$. This step is done by means of the first reconstruction theorem of Kontsevich and Manin or, equivalently, by the microscopic version of the DWVV equations.

3. We compute the contribution of reducible curves in $G[A_a^{N+1} \cap H, A_b^{N+1} \cap H, A_1^{1,N+1} \cap H, \cdots, A_1^{d-1,N+1} \cap H; d, N+1, k]$.

The 1st step gives next equalities, in writing them we use the convention that if number of insertion points gets lower than 3 then we insert an hyperplane condition and divide by the degree of the curve.

$$\begin{aligned}
& G[A_a^{N+1} \cap H, A_b^{N+1} \cap H; 1, N+1, k] \\
= \ & [A_{a+1}^{N+1}, A_{b+1}^{N+1}; 1, N+1, k] - [A_{a+b+1}^{N+1}; 1, N+1, k] \\
& (a+b = N-3+(N-k))
\end{aligned}$$

$$\begin{aligned}
& G[A_a^{N+1} \cap H, A_b^{N+1} \cap H, A_1^{N+1} \cap H; 2, N+1, k] \\
= \ & [A_{a+1}^{N+1}, A_{b+1}^{N+1}, A_2^{N+1}; 2, N+1, k] - [A_{a+2}^{N+1}, A_{b+1}^{N+1}; 2, N+1, k] \\
- \ & [A_{a+1}^{N+1}, A_{b+2}^{N+1}; 2, N+1, k] \\
- \ & [A_{a+b+1}^{N+1}, A_2^{N+1}; 2, N+1, k] + 2[A_{a+b+2}^{N+1}; 2, N+1, k] \\
& (a+b = N-3+2(N-k))
\end{aligned}$$

$$\begin{aligned}
& G[A_a^{N+1} \cap H, A_b^{N+1} \cap H, A_1^{N+1,1} \cap H, A_1^{N+1,2} \cap H; 3, N+1, k] \\
= \ & [A_{a+1}^{N+1}, A_{b+1}^{N+1}, A_2^{N+1,1}, A_2^{N+1,2}; 3, N+1, k]
\end{aligned}$$

$$
\begin{aligned}
&- \quad 2[A^{N+1}_{a+1}, A^{N+1}_{b+2}, A^{N+1}_{2}; 3, N+1, k] - 2[A^{N+1}_{a+2}, A^{N+1}_{b+1}, A^{N+1}_{2}; 3, N+1, k] \\
&- \quad [A^{N+1}_{a+b+1}, A^{N+1,1}_{2}, A^{N+1,2}_{2}; 3, N+1, k] - [A^{N+1}_{a+1}, A^{N+1}_{b+1}, A^{N+1}_{3}; 3, N+1, k] \\
&+ \quad 2[A^{N+1}_{a+1}, A^{N+1}_{b+3}; 3, N+1, k] + 2[A^{N+1}_{a+2}, A^{N+1}_{b+2}; 3, N+1, k] \\
&+ \quad 2[A^{N+1}_{a+3}, A^{N+1}_{b+1}; 3, N+1, k] \\
&+ \quad 4[A^{N+1}_{a+b+2}, A^{N+1}_{2}; 3, N+1, k] + [A^{N+1}_{a+b+1}, A^{N+1}_{3}; 3, N+1, k] \\
&- \quad 6[A^{N+1}_{a+b+3}; 3, N+1, k] \\
&\quad\quad (a+b = N-3+3(N-k)). \qquad (3.23)
\end{aligned}
$$

We assume now that the Fano index of X is at least 2, namely $N-k \geq 2$, then $a+b+1$ is greater than $N = dim(M^k_{N+1})+1$, and therefore (3.23) is truncated in an obvious way. At this point we recall the definition $[A^{N+1}_{a_1}, A^{N+1}_{a_1}, \cdots, A^{N+1}_{a_m}; d, N+1, k] = \langle \mathcal{O}_{e^{a_1}} \mathcal{O}_{e^{a_2}} \cdots, \mathcal{O}_{e^{a_m}} \rangle_{d, M^k_{N+1}, gr}$. In order to proceed we need to express $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^2} \rangle_{d, M^k_{N+1}, gr}$ and $\langle \mathcal{O}_{e^a} \mathcal{O}_{e_b} \mathcal{O}_{e^3} \rangle_{d, M^k_{N+1}, gr}$ in terms of $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e} \rangle_{d, M^k_{N+1}, gr}$. Our tool is the first reconstruction theorem of Kontsevich and Manin [21] it yields

$$
\begin{aligned}
\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^m} \rangle \langle \mathcal{O}_{e^{N-1-m}} \mathcal{O}_{e} \mathcal{O}_{e} \rangle &= \langle \mathcal{O}_{e^a} \mathcal{O}_{e} \mathcal{O}_{e^m} \rangle \langle \mathcal{O}_{e^{N-1-m}} \mathcal{O}_{e^b} \mathcal{O}_{e} \rangle \\
\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^m} \rangle \langle \mathcal{O}_{e^{N-1-m}} \mathcal{O}_{e^2} \mathcal{O}_{e} \rangle &= \langle \mathcal{O}_{e^a} \mathcal{O}_{e^2} \mathcal{O}_{e^m} \rangle \langle \mathcal{O}_{e^{N-1-m}} \mathcal{O}_{e^b} \mathcal{O}_{e} \rangle \\
\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^2} \mathcal{O}_{e^m} \rangle \langle \mathcal{O}_{e^{N-1-m}} \mathcal{O}_{e} \mathcal{O}_{e} \rangle &+ \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^m} \rangle \langle \mathcal{O}_{e^{N-1-m}} \mathcal{O}_{e^2} \mathcal{O}_{e} \mathcal{O}_{e} \rangle \\
= \langle \mathcal{O}_{e^a} \mathcal{O}_{e} \mathcal{O}_{e^2} \mathcal{O}_{e^m} \rangle \langle \mathcal{O}_{e^{N-1-m}} \mathcal{O}_{e^b} \mathcal{O}_{e} \rangle &+ \langle \mathcal{O}_{e^a} \mathcal{O}_{e} \mathcal{O}_{e^m} \rangle \langle \mathcal{O}_{e^{N-1-m}} \mathcal{O}_{e^b} \mathcal{O}_{e^2} \mathcal{O}_{e} \rangle
\end{aligned}
\qquad (3.24)
$$

We find in the end , if the Fano index of X is at least 2,

$$
\begin{aligned}
& \frac{1}{k} G[A^{N+1}_{N-2-m} \cap H, A^{N+1}_{m-1+(N-k)} \cap H; 1, N+1, k] \\
= \; & L^{N+1,k,1}_{m} := L^k_m \qquad (3.25) \\
& \frac{1}{k} G[A^{N+1}_{N-2-m} \cap H, A^{N+1}_{m-1+2(N-k)} \cap H, A^{N+1}_{1} \cap H; 2, N+1, k] \\
= \; & \frac{1}{2}(L^{N+1,k,2}_{m-1} + L^{N+1,k,2}_{m} + 2L^{N+1,k,1}_{m} \cdot L^{N+1,k,1}_{m+(N-k)}) \qquad (3.26) \\
& \frac{1}{k} G[A^{N+1}_{N-2-m} \cap H, A^{N+1}_{m-1+3(N-k)} \cap H, A^{N+1,1}_{1} \cap H, A^{N+1,2}_{1} \cap H; 3, N+1, k] \\
= \; & \frac{1}{6}(4L^{N+1,k,3}_{m-2} + 10L^{N+1,k,3}_{m-1} + 4L^{N+1,k,3}_{m} \\
& +12L^{N+1,k,2}_{m-1} \cdot L^{N+1,k,1}_{m+2(N-k)} + 12L^{N+1,k,2}_{m} \cdot L^{N+1,k,1}_{m+2(N-k)} \\
& +6L^{N+1,k,2}_{m} \cdot L^{N+1,k,1}_{m+1+2(N-k)} \\
& +6L^{N+1,k,2}_{m-1+(N-k)} \cdot L^{N+1,k,1}_{m-1} + 12L^{N+1,k,2}_{m-1+(N-k)} \cdot L^{N+1,k,1}_{m} \\
& +12L^{N+1,k,2}_{m+(N-k)} \cdot L^{N+1,k,1}_{m} \\
& +18L^{N+1,k,1}_{m} \cdot L^{N+1,k,1}_{m+(N-k)} \cdot L^{N+1,k,1}_{m+2(N-k)}) \qquad (3.27)
\end{aligned}
$$

We make the hypothesis that the Schubert varieties of conics and lines on Y and on X which are associated with the given linear spaces S_i, T_j and their intersections have the right dimension. A count of dimensions shows that for the cases of degree 1 and degree 2 there is no contribution from reducible curves and therefore

$G[A^{N+1}_{N-2-m} \cap H, A^{N+1}_{m-1+(N-k)} \cap H; 1, N+1, k]/k = L^{N,k,1}_{m}$

and

$G[A^{N+1}_{N-2-m} \cap H, A^{N+1}_{m-1+(N-k)} \cap H, A^{N+1}_{1} \cap H; 2, N+1, k]/k = L^{N,k,2}_{m}$.

For cubic curves there is a contribution due to reducible connected curves which are made of a line lying on X and of a conic lying on Y. There are two cases which occur, in one instance the line meets $A^{N+1}_{N-2-m}\cap H$, the conic meets the line and $A^{N+1}_{m-1+3(N-k)}\cap H$. In the other case the incidence conditions with the linear spaces are reversed. In this way

$$\begin{aligned}
& G[A^{N+1}_{N-2-m}\cap H, A^{N+1}_{m-1+3(N-k)}\cap H, A^{N+1,1}_{1}\cap H, A^{N+1,2}_{1}\cap H; 3, N+1, k]/k \\
= \;& 3L^{N,k,3}_{m} + R\frac{1}{k} \\
= \;& 3L^{N,k,3}_{m} + \frac{1}{2}L^{N+1,k,2}_{m}\cdot L^{N,k,1}_{m+2(N-k)} + \frac{1}{2}L^{N+1,k,2}_{m-1+(N-k)}\cdot L^{N,k,1}_{m}
\end{aligned} \tag{3.28}$$

Q.E.D

We come now to the case of Fano index 1. The same type of computations as above yield

Theorem 3 *If X is a hypersurface of degree k in CP^k the recursion relations for the basic invariants of conics and cubic curves are the same as given in Theorem 1, instead the numbers of lines satisfy the law*

$$L^{k+1,k,1}_{m} = L^{k+2,k,1}_{m} - L^{k+2,k,1}_{0} = L^{k+2,k,1}_{m} - k! \tag{3.29}$$

Proof In this case, $a+b+1 = N-2+d$, from (3.23).

$$\begin{aligned}
& [A^{N+1}_{a}\cap H, A^{N+1}_{b}\cap H; 1, N+1, k] \\
= \;& [A^{N+1}_{a+1}, A^{N+1}_{b+1}; 1, N+1, k] - [A^{N+1}_{N-1}; 1, N+1, k] \\
& (a+b+1 = N-1)
\end{aligned} \tag{3.30}$$

Now we can prove:

Corollary 1 *The main relation of the quantum ring of a Fano hypersurface M^k_N with index $N-k\geq 2$ is of the form*

$$(\mathcal{O}_e)^{N-1} - k^k(\mathcal{O}_e)^{k-1}q = 0 \tag{3.31}$$

up to q^3.

Proof The recursion relations of Theorem 2. do not change $\gamma^{N,k,d}_0$, namely $\gamma^{N,k,d}_0 = \gamma^{N+1,k,d}_0$. If $N\geq 2k+1$, then $\gamma^{N,k,d}_0 = 0 (d\geq 2)$, because of the vanishing conditions due to the topological selection rule.On the other hand $\gamma^{N,k,1}_0 = k^k$, from Schubert calculus cf. [1].

Corollary 2 *The main relation of the quantum cohomology ring of a Fano hypersurface of index 1 and dimension $k-1$ is of the form*

$$(\mathcal{O}_e + k!q)^k - k^k(\mathcal{O}_e + k!q)^{k-1}q = 0 \tag{3.32}$$

up to q^3.

Proof

Consider the multiplication rule (2.7):

$$\mathcal{O}_e\cdot\mathcal{O}_{e^{k-1-m}} := \mathcal{O}_{e^{k-m}} + qL^{k+1,k,1}_{m}\mathcal{O}_{e^{k-1-m}} + \sum_{d=2}^{\infty} q^d L^{k+1,k,d}_{m}\mathcal{O}_{e^{k-m-d}} \tag{3.33}$$

This gives:

$$\begin{aligned}
(\mathcal{O}_e + k!q)\cdot\mathcal{O}_{e^{k-1-m}} &= \mathcal{O}_{e^{k-m}} + q(L^{k+1,k,1}_{m} + k!)\mathcal{O}_{e^{k-1-m}} + \sum_{d=2}^{\infty} q^d L^{k+1,k,d}_{m}\mathcal{O}_{e^{k-m-d}} \\
&= \mathcal{O}_{e^{k-m}} + \sum_{d=1}^{\infty} q^d \tilde{L}^{k+1,k,d}_{m}\mathcal{O}_{e^{k-m-d}}.
\end{aligned} \tag{3.34}$$

Set now $F = \mathcal{O}_e + k!q$, use F as a multiplicative generator and write

$$\mathcal{O}_{e^{k-m}} = (F)^{k-m} - \sum_{d=1}^{\infty} q^d \tilde{\gamma}_m^{k+1,k,d} (F)^{k-m-d} \quad (m = 0, 1, \cdots, k-1) \tag{3.35}$$

A standard computation yields $\tilde{\gamma}_0^{k+1,k,d} = \gamma_0^{k+2,k,d}$, and we conclude by descending induction as in the proof of the preceding corollary gives the wished relation.

Our last result is also easily proved by descending induction on N

Corollary 3 *If $d \leq 3$ and $N - k \geq 1$) the structure constants $L_m^{N,k,d}$ can be written as a homogeneous polynomial of degree d in the structure constants of lines L_m^k*

4 Recursive formulas in the Calabi-Yau case.

Here we try to understand how the recursive formulas change when the hypersurface becomes of Calabi-Yau type, i.e. when we deal with M_k^k of degree k in CP^{k-1}. In this situation we can proceed as before for lines and conics. On the other hand we cannot use the same method for curves of degree 3, because it is difficult in this case to control the contribution from reducible curves . We give instead a conjectural recursive formulas for cubics. In the last part of the section we explain the trend of thought which led us to the conjecture.

We recall first that given a general point on M_k^k there are no rational curves meeting it, and therefore $L_0^{k,k,d} = 0$.

Theorem 4 *The recursive laws for lines and conics on a Calabi-Yau hypersurface of degree k are for $m \geq 1$*

$$L_m^{k,k,1} = L_m^{k+1,k,1} - L_1^{k+1,k,1} \tag{4.36}$$

$$\begin{aligned} L_m^{k,k,2} &= \frac{1}{2}(L_{m-1}^{k+1,k,2} + L_m^{k+1,k,2} - L_0^{k+1,k,2} - L_1^{k+1,k,2} \\ &\quad +2(L_m^{k+1,k,1} - L_1^{k+1,k,1})^2) \end{aligned} \tag{4.37}$$

Proof of Theorem 4

We go back to the specialization formula (3.23), which we use with the condition $a + b = N - 3$ because now we have $N = k$. Repeated use of the first reconstruction theorem yields

$$\begin{aligned} & \frac{1}{k} G[A_{k-2-m}^{k+1} \cap H, A_{m-1}^{k+1} \cap H; 1, k+1, k] \\ = \;& L_m^{k+1,k,1} - L_1^{k+1,k,1} \end{aligned} \tag{4.38}$$

$$\begin{aligned} & \frac{1}{k} G[A_{k-2-m}^{k+1} \cap H, A_{m-1}^{k+1} \cap H, A_1^{k+1} \cap H; 2, k+1, k] \\ = \;& \frac{1}{2}(L_{m-1}^{k+1,k,2} + L_m^{k+1,k,2} - L_0^{k+1,k,2} - L_1^{k+1,k,2} \\ + \;& 2L_m^{k+1,k,1} \cdot (L_m^{k+1,k,1} - L_1^{k+1,k,1})) \end{aligned} \tag{4.39}$$

Next we check the contribution from reducible curves. For lines there are no reducible curves, so that

$$\frac{1}{k} G[A_{k-2-m}^{k+1} \cap H, A_{m-1}^{k+1} \cap H; 1, k+1, k] = L_m^{k,k,1}. \tag{4.40}$$

In case of conics, the reducible curves are given by two intersecting lines, one lying on $M_k^k = M_{k+1}^k \cap H$ and the other on M_{k+1}^k, hence it is

$$\frac{1}{k} G[A_{k-2-m}^{k+1} \cap H, A_{m-1}^{k+1} \cap H, A_1^{k+1} \cap H; 2, k+1, k] = L_m^{k,k,2} + L_1^{k+1,k,1} \cdot L_m^{k,k,1} \tag{4.41}$$

Q.E.D.

Next we deal with cubic curves, the Calabi-Yau condition implies again $L_m^{k,k,3} = 0$ for $0 \leq m \leq 1$, our proposal for larger m is the following:

Conjecture 1 *The recursive law for curves of degree* 3 *on* M_k^k *and for* $m \geq 2$ *becomes:*

$$\begin{aligned}
L_m^{k,k,3} \quad = \quad & \frac{1}{18}(4L_{m-2}^{k+1,k,3} + 10L_{m-1}^{k+1,k,3} + 4L_m^{k+1,k,3} \\
& -10L_0^{k+1,k,3} - 4L_1^{k+1,k,3} \\
& +12L_{m-1}^{k+1,k,2} \cdot L_m^{k+1,k,1} + 12L_m^{k+1,k,2} \cdot L_m^{k+1,k,1} \\
& +6L_m^{k+1,k,2} \cdot L_{m+1}^{k+1,k,1} \\
& +6L_{m-1}^{k+1,k,2} \cdot L_{m-1}^{k+1,k,1} + 12L_{m-1}^{k+1,k,2} \cdot L_m^{k+1,k,1} \\
& +12L_m^{k+1,k,2} \cdot L_m^{k+1,k,1} \\
& +18(L_m^{k+1,k,1} - L_1^{k+1,k,1})^2 \cdot (L_m^{k+1,k,1} + 2L_1^{k+1,k,1})) \\
& -\frac{1}{6}L_{m-1}^{k+1,k,2} \cdot L_m^{k+1,k,1} - \frac{1}{6}L_m^{k+1,k,2} \cdot L_m^{k+1,k,1} \\
& -\frac{3}{4}L_0^{k+1,k,2} \cdot L_m^{k+1,k,1} - \frac{3}{4}L_1^{k+1,k,2} \cdot L_m^{k+1,k,1} \\
& -\frac{5}{12}L_1^{k+1,k,2} \cdot L_1^{k+1,k,1} - \frac{5}{12}L_0^{k+1,k,2} \cdot L_1^{k+1,k,1} \\
& -\frac{1}{3}L_1^{k+1,k,2} \cdot L_2^{k+1,k,1} \\
& -3L_1^{k+1,k,1} \cdot \frac{1}{2}(L_{m-1}^{k+1,k,2} + L_m^{k+1,k,2} - L_0^{k+1,k,2} - L_1^{k+1,k,2} \\
& +2(L_m^{k+1,k,1} - L_1^{k+1,k,1})^2).
\end{aligned} \tag{4.42}$$

We came to this formula by means of the following considerations. Using Theorem 4 one can compute the main relation of $H_{q,e}^*(M_k^k)$ up to degree 2, this reads:

$$\begin{aligned}
& (1 - (k^k - (k-2)L_1^k - 2L_0^k)q \\
& -(2k^k L_0^k + (k-3)k^k L_1^k - 3(L_0^k)^2 - (2k-6)L_1^k L_0^k - \frac{(k-3)(k-2)}{2}(L_1^k)^2 \\
& -\frac{k}{2}L_0^{k+1,k,2} - \frac{k-2}{2}L_1^{k+1,k,2})q^2 - \cdots)(\mathcal{O}_e)^{k-1} = 0
\end{aligned} \tag{4.43}$$

On the other hand one has from [18] and [16] that the main relation can be written using the $k-2$ point correlation function of the pure matter theory in the form

$$\frac{k}{\langle \prod_{j=1}^{k-2} \mathcal{O}_e(z_j) \rangle_{M_k^k, matter}} (\mathcal{O}_e)^{k-1} = 0 \tag{4.44}$$

and that it is

$$\begin{aligned}
\langle \prod_{j=1}^{k-2} \mathcal{O}_e(z_j) \rangle_{M_k^k, matter} \quad = \quad & k + k^{k+1}(1 - 2a_1 - (k-2)(b_1))q \\
+ \quad & k^{2k+1}(1 - 2a_1 - b_1 + 3(a_1)^2 - 2a_2 + 2a_1 \cdot b_1 \\
+ \quad & (k-2)(-b_1 + 4a_1 \cdot b_1 + 2(b_1)^2 - 2b_2) \\
+ \quad & \frac{(k-2)(k-3)}{2}(b_1)^2)q^2 + \cdots
\end{aligned} \tag{4.45}$$

Here

$$a_d = \frac{(kd)!}{(d!)^k k^{kd}} \ , \quad b_d = a_d(\sum_{i=1}^{d}\sum_{m=1}^{k-1}\frac{m}{i(ki-m)})$$

are the coefficients of the hypergeometric series associated to the solutions

$$W_0(x) = \sum_{d=0}^{\infty} a_d e^{dx}, \quad W_1(x) = \sum_{d=1}^{\infty} b_d e^{dx} + W_0(x)x$$

of the differential equation

$$((\frac{d}{dx})^{k-1} - e^x(\frac{d}{dx}+\frac{1}{k})(\frac{d}{dx}+\frac{2}{k})\cdots(\frac{d}{dx}+\frac{k-1}{k}))W_i(x) = 0 \tag{4.46}$$

By comparison of (4.43) with (4.45), we notice the following equalities:

$$\begin{aligned} k^k a_1 &= L_0^k \\ k^k b_1 &= L_1^k \\ k^{2k} a_2 &= \frac{1}{2}(L_0^{k+1,k,2} + 2(L_0^k)^2) \\ k^{2k} b_2 &= \frac{1}{4}(L_1^{k+1,k,2} + L_0^{k+1,k,2} + 2L_1^k L_1^k) + L_1^k L_0^k. \end{aligned} \tag{4.47}$$

These equalities can be organized more systematically by means of generating functions. To this aim we need :

Definition 1 *For arbitrary N and k let $\tilde{L}_m^{N,k,d}$ be the integer obtained by applying formally the recursive laws of Theorem 2.*

Remark. One should note: (i) $\tilde{L}_i^{k,k}(e^x) = \tilde{L}_{k-1-i}^{k,k}(e^x)$, (ii) if the index $N-k$ is at least 2 then $\tilde{L}_m^{N,k,d}$ must be the ordinary structural constant $L_m^{N,k,d}$ of the Fano hypersurface.

Now we can rewrite (4.47) as

$$\begin{aligned} k^k a_1 &= \tilde{L}_0^{k,k,1} \\ k^k b_1 &= \tilde{L}_1^{k,k,1} \\ k^{2k} a_2 &= \tilde{L}_0^{k,k,2} \\ k^{2k} b_2 &= \frac{1}{2}\tilde{L}_1^{k,k,2} + \tilde{L}_1^{k,k,1}\cdot\tilde{L}_0^{k,k,1} \end{aligned} \tag{4.48}$$

After having performed some numerical computations, we have noticed that also the following relations should hold true:

$$\begin{aligned} k^{3k} a_3 &= \tilde{L}_0^{k,k,3} \\ k^{3k} b_3 &= \frac{1}{3}\tilde{L}_1^{k,k,3} + \frac{1}{2}\tilde{L}_1^{k,k,2}\cdot\tilde{L}_0^{k,k,1} + \tilde{L}_1^{k,k,1}\cdot\tilde{L}_0^{k,k,2}. \end{aligned} \tag{4.49}$$

Consider next the generating functions:

$$\tilde{L}_i^{k,k}(\tilde{q}) := 1 + \sum_{d=1}^{\infty}\tilde{L}_i^{k,k,d}\tilde{q}^d \ , \quad \tilde{q} := e^x \tag{4.50}$$

and define

$$t := x + (\sum_{j=1}^{\infty} b_j k^{kj} e^{jx})/(\sum_{j=0}^{\infty} a_j k^{kj} e^{jx}). \tag{4.51}$$

The preceding equalities motivate us to expect:

$$\tilde{L}_0^{k,k}(\tilde{q}) = \sum_{j=0}^{\infty} a_j k^{kj} e^{jx}, \quad \tilde{L}_1^{k,k}(e^x) = \frac{dt}{dx}. \tag{4.52}$$

We use the virtual structural constants $\tilde{L}$ to define a virtual quantum product determined by the action of a ring generator G which operates according to the rules:

$$\begin{aligned} G \cdot \mathcal{O}_{e^{m-1}} &= \tilde{L}_{k-1-m}^{k,k}(e^x)\mathcal{O}_{e^m} \qquad (4.53) \\ & \quad (m = 1, 2, \cdots, k-2) \\ G \cdot \mathcal{O}_{e^{k-2}} &= 0 \quad . \qquad (4.54) \end{aligned}$$

We note the relation $G = G \cdot 1 = \tilde{L}_{k-2}^{k,k}(e^x)\mathcal{O}_e$. We expect that the structure constants of the virtual action satisfy the following equality, which in fact may be checked up to $\tilde{q}^3$ using the recursive laws for the Fano case:

$$\prod_{i=0}^{k-1} \tilde{L}_i^{k,k}(e^x) \quad = \quad (1 - k^k e^x)^{-1} \tag{4.55}$$

and this yields the relation:

$$(1 - k^k e^x) \cdot (\tilde{L}_0^{k,k}(e^x))^2 \cdot (G)^{k-1} = 0 \tag{4.56}$$

On the other hand the *true* quantum cohomology ring satisfies a similar multiplication rule:

$$\begin{aligned} \mathcal{O}_e \cdot 1 &= \mathcal{O}_e \\ \mathcal{O}_e \cdot \mathcal{O}_{e^{m-1}} &= L_{k-1-m}^{k,k}(e^t)\mathcal{O}_{e^m} \quad (m = 2, 3, \cdots, k-2) \\ \mathcal{O}_e \cdot \mathcal{O}_{e^{k-3}} &= \mathcal{O}_{e^{k-2}} \\ \mathcal{O}_e \cdot \mathcal{O}_{e^{k-2}} &= 0 \qquad (4.57) \\ \text{where} & \\ L_i^{k,k}(e^t) &:= 1 + \sum_{d=1}^{\infty} L_i^{k,k,d} e^{dt}. \qquad (4.58) \end{aligned}$$

We can compute $L_i^{k,k}(e^t)$ in concrete examples using the method of torus localization see [17] for details and results.

Now we search for a transformation rule to pass from the virtual to the true quantum multiplication. To this aim we find useful to introduce a formal definition:

Definition 2 *The commutative product* $(*)$ *between differential operators of the form* $f(x)\frac{d^m}{dx^m}$ *is given by:*

$$(f(x)\frac{d^m}{dx^m}) * (g(x)\frac{d^n}{dx^n}) = (f(x) \cdot g(x))\frac{d^{m+n}}{dx^{m+n}}. \tag{4.59}$$

Given the coordinate change $x = x(t)$ *we define a map from* $\frac{d}{dx}$ *operators to* $\frac{d}{dt}$ *operators by means of the rule*

$$f(x)\frac{d^m}{dx^m} \to f(x(t))(\frac{dt}{dx})^m \frac{d^m}{dt^m}. \tag{4.60}$$

At this point we can relate the quantum products laws using as an intermediate step the product of differential operators. To start we propose the correspondence

$$\mathcal{O}_e = \frac{d}{dt}, \quad G = \frac{d}{dx}. \tag{4.61}$$

Then one has

$$\begin{aligned}
\frac{d}{dx} * \mathcal{O}_{e^{m-1}} &= \tilde{L}^{k,k}_{k-1-m}(e^x)\mathcal{O}_{e^m}, \quad (m = 1, 2, \cdots, k-2) \\
\frac{d}{dx} * \mathcal{O}_{e^{k-2}} &= 0,
\end{aligned} \tag{4.62}$$

and

$$\begin{aligned}
\frac{d}{dt} * 1 &= \mathcal{O}_e \\
\frac{d}{dt} * \mathcal{O}_{e^{m-1}} &= L^{k,k}_{k-1-m}(e^t)\mathcal{O}_{e^m} \quad (m = 2, 3, \cdots, k-2) \\
\frac{d}{dt} * \mathcal{O}_{e^{k-3}} &= \mathcal{O}_{e^{k-2}}, \quad \frac{d}{dt} * \mathcal{O}_{e^{k-2}} = 0.
\end{aligned} \tag{4.63}$$

It follows from (4.62):

$$\frac{d}{dx} * 1 = \tilde{L}^{k,k}_{k-2}(e^x) \cdot \mathcal{O}_e = \tilde{L}^{k,k}_{k-2}(e^x) \cdot \frac{d}{dt} = \frac{dt}{dx} \cdot \frac{d}{dt} \tag{4.64}$$

This equality suggests that (4.62) and (4.63) become isomorphic if we use the transformation of differential operators defined above. Compare now the coefficients for $\mathcal{O}_{e^\alpha}$ in (4.62) with (4.63), then the wished isomorphism yields the equality

$$\prod_{j=1}^{\alpha} (\tilde{L}^{k,k}_{k-1-j}(e^{x(t)}) \frac{dx}{dt}) = \prod_{j=1}^{\alpha} L^{k,k}_{k-1-j}(e^t). \tag{4.65}$$

We find in this way the transformation laws that we were looking for, they are:

$$\frac{\tilde{L}^{k,k}_i(e^{x(t)})}{\tilde{L}^{k,k}_1(e^{x(t)})} = \tilde{L}^{k,k}_i(e^{x(t)}) \frac{dx}{dt} = L^{k,k}_i(e^t). \tag{4.66}$$

This is the rule that provides the recursive formulas for curves of arbitrary degree d on the Calabi-Yau hypersurface M^k_k once that we know the recursive formulas for curves up to degree d on Fano hypersurfaces. At this point we obtain the recursive formulas for cubics in Conjecture 1 by means of elementary calculations.

Example The true quantum cohomology ring of the quintic Calabi-Yau threefold is:

$$\begin{aligned}
\mathcal{O}_e \cdot 1 &= \mathcal{O}_e \\
\mathcal{O}_e \cdot \mathcal{O}_e &= \mathcal{O}_{e^2}(1 + 575e^t + 975375e^{2t} + 1712915000e^{3t} + \cdots) \\
\mathcal{O}_e \cdot \mathcal{O}_{e^2} &= \mathcal{O}_{e^3} \\
\mathcal{O}_e \cdot \mathcal{O}_{e^3} &= 0.
\end{aligned} \tag{4.67}$$

while the associated virtual ring is:

$$\begin{aligned}
G \cdot 1 &= \mathcal{O}_e(1 + 770e^x + 1435650e^{2x} + 3225308000e^{3x} + \cdots) \\
G \cdot \mathcal{O}_e &= \mathcal{O}_{e^2}(1 + 1345e^x + 3296525e^{2x} + 8940963625e^{3x} + \cdots) \\
G \cdot \mathcal{O}_{e^2} &= \mathcal{O}_{e^3}(1 + 770e^x + 1435650e^{2x} + 3225308000e^{3x} + \cdots) \\
G \cdot \mathcal{O}_{e^3} &= 0
\end{aligned} \tag{4.68}$$

Using (4.66) we find that:

$$575 = 1345 - 770 \tag{4.69}$$

$$
\begin{aligned}
975375 &= 3296525 - 1435650 + 770 \cdot 770 \\
&\quad -1345 \cdot 770 - 770 \cdot (1345 - 770) &(4.70)\\
1712915000 &= 8940963625 - 3225308000 \\
&\quad +2 \cdot 770 \cdot 1435650 - 770^3 + 1345 \cdot (770)^2 \\
&\quad -1345 \cdot 1435650 - 3296525 \cdot 770 \\
&\quad -2 \cdot 770 \cdot (3296525 - 1435650 + 770 \cdot 770 - 1345 \cdot 770) \\
&\quad +(\frac{3}{2} \cdot (770)^2 - \frac{1}{2} \cdot 1435650) \cdot (1345 - 770). &(4.71)
\end{aligned}
$$

5 On the construction of the recursive formulas for rational curves of larger degree.

Motivated by the preceding results, we begin this section by proposing some conjectures. They are strong enough to allow in principle the construction of the expected recursive formulas for curves of higher degree, and we explicitly produce the law for $d = 4, 5$.

Conjecture 2 *There are universal recursive polynomial laws which express the structure constants $L_m^{N,k,d}$ on a Fano variety in terms of $L_m^{N+1,k,n}$ $(1 \leq n \leq d)$. The formulas have the following properties.*

1. The form of the recursive polynomials is invariant if the index $N - k \geq 2$, and the equality $\gamma_0^{N,k,d} = \gamma_0^{N+1,k,d}$ for the coefficients in the fundamental relations is a consequence of them.

2. If $N - k = 1$ the recursive formulas change only for the case of lines, i.e. $d = 1$.

We keep the notations of section 4, so that $\tilde{L}_m^{N,k,d}$ represents the result of a formal iteration of the recursive functions for fixed k down to any chosen N, and then $\tilde{L}_m^{N,k}$ is the associated generating function.

Conjecture 3 *Formal iteration of the laws of Conjecture 2 for descending N down to the case $N = k$ yields the coefficients of the hypergeometric series used in the mirror calculation, i.e., it should be*

$$
\begin{aligned}
k^{dk} a_d &= \tilde{L}_0^{k,k,d} \\
k^{dk} b_d &= \frac{1}{d} \tilde{L}_1^{k,k,d} + \sum_{m=1}^{d-1} \frac{1}{m} \tilde{L}_1^{k,k,m} \cdot \tilde{L}_0^{k,k,d-m},
\end{aligned} \tag{5.72}
$$

and the same procedure gives the structure constants of the quantum cohomology ring of the Calabi-Yau hypersurface of degree k, using the rule

$$
L_i^{k,k}(e^t) = \frac{\tilde{L}_i^{k,k}(e^{x(t)})}{\tilde{L}_1^{k,k}(e^{x(t)})}. \tag{5.73}
$$

Remark. It is an immediate consequence of the conjectures and of the vanishing conditions of section 2 that the structural constant $L_m^{N,k,d}$ is a polynomial of degree d in the constants L_m^k, $N \geq k$.

We proceed now to the construction of the recursive formulas for the case $d = 4$. Our method is based on the expectation that the specialization procedure gives if not the right coefficients at least the right monomials which appear in the recursive laws. We formalize this below with a conjecture.

We start by constructing some technical formulas for the factorization of the Gromov-Witten invariants.

Let $\{n_*\}:=\{n_1,n_2,\cdots,n_l\}$ and $ind(\{n_*\})=\sum_{j=1}^{l}(n_j-1)$. We formally define $ind(\{\emptyset\})$ to be 0. We have a formula for the correlation functions (Gromov-Witten invariants) of the topological sigma model on M^k_{N+1} coupled to gravity, which reads:

$$\begin{aligned}
& k^{-1}\langle\cdot\mathcal{O}_{e^a}\mathcal{O}_{e^{n_1}}\mathcal{O}_{e^{n_2}}\cdots\mathcal{O}_{e^{n_l}}\mathcal{O}_{e^b}\rangle_{d,M^k_{N+1},gr} \\
= & \sum_{d_1=0}^{d}\sum_{d_2=0}^{d_1}\cdots\sum_{d_{ind(\{n_*\})}=0}^{d_{ind(\{n_*\})-1}} C^d(\{n_*\};d_1,d_2,\cdots,d_{ind(\{n_*\})})\prod_{i=0}^{ind(\{n_*\})} L^{N+1,k,d_i-d_{i+1}}_{n+1-a-i+(N-k+1)(d-d_i)}
\end{aligned} \tag{5.74}$$

$$d_0:=d,\quad N-k+1\geq ind(\{n_*\})+1 \tag{5.75}$$

The coefficients $C^d(\{n_*\};d_1,\cdots,d_{ind(\{n_*\})})$ which appear here have the following properties:

$$C^d(\{m\};d_1,\cdots,d_{m-1})=1 \tag{5.76}$$

$$C^d(\{n_*\}\cup\{1\};d_1,\cdots,d_{ind(\{n_*\})})=dC^d(\{n_*\};d_1,\cdots,d_{ind(\{n_*\})}) \tag{5.77}$$

One can determine $C^d(\{n_*\};d_1,\cdots,d_{ind(\{n_*\})})$ by means of the recursive relation,

$$\begin{aligned}
& C^d(\{n_*\};d_1,\cdots,d_{ind(\{n_*\})}) \\
= & \sum_{\substack{\{l_*\}\amalg\{m_*\}=\{n_*\}/\{n_l\}\\ \{m_*\}\neq\emptyset}} (C^{d-d_{ind(\{l_*\})+n_l-1}}(\{l_*\}\cup\{n_l-1\};d_1-d_{ind(\{l_*\})+n_l-1}, \\
& \cdots,d_{ind(\{l_*\})+n_l-2}-d_{ind(\{l_*\})+n_l-1})\cdot \\
& C^{d_{ind(\{l_*\})+n_l-1}}(\{m_*\};d_{1+ind(\{l_*\})+n_l-1},\cdots,d_{ind(\{n_*\})})\cdot d_{ind(\{l_*\})+n_l-1}) \\
+ & C^{d-d_{ind(\{n_*\})}}(\{n_*\}/\{n_l\}\cup\{n_l-1\};d_1-d_{ind(\{n_*\})},\cdots,d_{ind(\{n_*\})-1}-d_{ind(\{n_*\})}).
\end{aligned} \tag{5.78}$$

Proof

We prove (5.78) by induction of $ind(\{n_*\})$. We denote $\mathcal{O}_{e^{n_1}}\mathcal{O}_{e^{n_2}}\cdots\mathcal{O}_{e^{n_l}}$ as $\mathcal{O}_{e^{\{n_*\}}}$ for brevity. The first reconstruction theorem of KM yields:

$$\begin{aligned}
& \sum_{\substack{\{l_*\}\amalg\{m_*\}\\=\{n_*\}/\{n_l\}}}\sum_{d_0=0}^{d} && k^{-1}\langle\mathcal{O}_{e^a}\mathcal{O}_{e^{\{l_*\}}}\mathcal{O}_{e^b}\mathcal{O}_{e^{n_l+ind(\{m_*\})-(N-k+1)d_0}}\rangle_{d-d_0,gr}\cdot \\
& && \cdot k^{-1}\langle\mathcal{O}_{e^{ind(\{l_*\})-(N-k+1)(d-d_0)+a+b}}\mathcal{O}_{e^{\{m_*\}}}\mathcal{O}_{e^{n_l-1}}\mathcal{O}_{e}\rangle_{d_0,gr} \\
= & \sum_{\substack{\{l_*\}\amalg\{m_*\}\\=\{n_*\}/\{n_l\}}}\sum_{d_0=0}^{d} && k^{-1}\langle\mathcal{O}_{e^a}\mathcal{O}_{e^{\{l_*\}}}\mathcal{O}_{e^{n_l-1}}\mathcal{O}_{e^{b+1+ind(\{m_*\})-(N-k+1)d_0}}\rangle_{d-d_0,gr}\cdot \\
& && \cdot k^{-1}\langle\mathcal{O}_{e^{a-1+n_l+ind(\{l_*\})-(N-k+1)(d-d_0)}}\mathcal{O}_{e^{\{m_*\}}}\mathcal{O}_{e}\mathcal{O}_{e^b}\rangle_{d_0,gr}.
\end{aligned} \tag{5.79}$$

The l.h.s. of (5.79) has the contribution of $d_0=0$ and $\{m_*\}=\{\emptyset\}$ because $ind(\{n_*\})-(N-k+1)d_0\leq -1$ and because the classical correlation function remains non-zero only if the number of operator insertion point equals 3. Then we have

$$(\text{the l.h.s.})\text{ of }(5.79)\quad=\quad k^{-1}\langle\mathcal{O}_{e^a}\mathcal{O}_{e^{\{n_*\}}}\mathcal{O}_{e^b}\rangle_{d,gr}. \tag{5.80}$$

On the other hand, we can rewrite the r.h.s. of (5.79) from the assumption of induction,

$$(\sum_{\substack{\{l_*\}\amalg\{m_*\}=\{n_*\}/\{n_l\}\\ \{m_*\}\neq\emptyset}}\sum_{d_0=0}^{d}\sum_{t_1=0}^{d-d_0}\cdots\sum_{t_{ind(\{l_*\})+n_l-2}=0}^{t_{ind(\{l_*\})+n_l-3}} C^{d-d_0}(\{l_*\}\cup\{n_l-1\};t_1,\cdots,t_{ind(\{l_*\})+n_l-2})\cdot$$

$$\prod_{i=0}^{ind(\{l_*\})+n_l-2} L^{N+1,k,t_i-t_{i+1}}_{n+1-a-i+(N-k+1)(d-d_0-t_i)}) \cdot$$
$$(\sum_{u_1=0}^{d_0}\cdots\sum_{u_{ind(\{l_*\})}=0}^{u_{ind(\{l_*\})-1}} d_0C^{d_0}(\{l_*\}\cup\{n_l-1\};u_1,\cdots,u_{ind(\{m_*\})})\cdot$$
$$\prod_{j=0}^{ind(\{m_*\})} L^{N+1,k,u_j-u_{j+1}}_{n+1-a+1+ind(\{l_*\})-j+(N-k+1)(d-d_0)+(N-k+1)(d_0-u_j)})$$
$$+\sum_{\substack{\{l_*\}\amalg\{m_*\}=\{n_*\}/\{n_l\}\\ \{m_*\}\neq\emptyset}}\sum_{d_0=0}^{d}\sum_{t_1=0}^{d-d_0}\cdots\sum_{t_{ind(\{n_*\})-1}=0}^{t_{ind(\{n_*\})-2}} C^{d-d_0}(\{n_*\}/\{n_l\}\cup\{n_l-1\};t_1,\cdots,t_{ind(\{n_*\})-1})\cdot$$
$$\prod_{i=0}^{ind(\{n_*\})-1} L^{N+1,k,t_i-t_{i+1}}_{n+1-a-i+(N-k+1)(d-d_0-t_i)}\cdot L^{N+1,k,t_i-t_{i+1}}_{n+1-a-ind(\{n_*\})+(N-k+1)(d-d_0)} \tag{5.81}$$

Then an appropriate change of t_i's and u_i's leads to (5.78). Q.E.D.

This formula tells us that if we take $N-k$ fairly large, we can determine the form of recursive formula without subtle complexity. Our conjecture asserts that these formulas obtained works in $N-k\geq 2$ case for rational curves of arbitrary degree and in $N-k=1$ case for curves whose degree is more than 2.

Using this, we calculate some examples.

$$\begin{aligned}
&C^d(\{2,2\};d_1,d_2)=d_1+d-d_2\\
&C^d(\{3,2\};d_1,d_2,d_3)=d_1+d-d_3\\
&C^d(\{3,3\};d_1,d_2,d_3,d_4)=d+d_1+d_2-d_3-d_4\\
&C^d(\{4,2\};d_1,d_2,d_3,d_4)=d+d_1-d_4\\
&C^d(\{2,2,2\};d_1,d_2,d_3)=2d_2\cdot(d-d_2)+d_1\cdot(d_1+d_2-d_3)\\
&+(d-d_3)\cdot(d+d_1-d_2-d_3)\\
&C^d(\{3,2,2\};d_1,d_2,d_3,d_4)=d_1\cdot(d_1+d_2-d_4)+(d-d_2)\cdot d_2+(d-d_3)\cdot d_3\\
&+(d-d_4)\cdot(d+d_1-d_3-d_4)\\
&C^d(\{2,2,2,2\};d_1,d_2,d_3,d_4)=d_1\cdot(2d_3\cdot(d_1-d_3)+d_2\cdot(d_2+d_3-d_4)\\
&+(d_1-d_4)\cdot(d_1+d_2-d_3-d_4))+3(d-d_2)\cdot d_2\cdot(d_2+d_3-d_4)\\
&+3(d-d_3)\cdot d_3\cdot(d+d_1-d_2-d_3)+(d-d_4)\cdot(2(d_2-d_4)\cdot(d-d_2)\\
&+(d_1-d_4)\cdot(d_1+d_2-d_3-d_4)+(d-d_3)\cdot(d+d_1-d_2-d_3))
\end{aligned} \tag{5.82}$$

And specialization process can be systematically done by the following formula.

$$\begin{aligned}
&[A_{a_1-1}\cap H,A_{a_2-1}\cap H,\cdots,A_{a_{d+1}-1}\cap H;d,N+1,k]\\
&=\sum_{m=1}^{d+1}\sum_{\substack{\amalg_{j=1}^m U_j=\{a_*\}\\ U_j\neq\emptyset}}(-1)^{d+1-m}([A_{ind(U_1)+1},A_{ind(U_2)+1},\cdots,A_{ind(U_m)+1};d,N+1,k]\cdot\\
&\cdot(\prod_{j=1}^{m}(^\sharp(U_j)-1)!))
\end{aligned} \tag{5.83}$$

Application of (5.83) leads us to,

$$G[A_a^{N+1}\cap H,A_b^{N+1}\cap H,A_1^{N+1}\cap H,A_1^{N+1}\cap H,A_1^{N+1}\cap H;4,N+1,k]$$

$$
\begin{aligned}
&= [A_{a+1}^{N+1}, A_{b+1}^{N+1}, A_2^{N+1}, A_2^{N+1}A_2^{N+1}; 4, N+1, k] \\
&- 3([A_{a+1}^{N+1}, A_{b+2}^{N+1}, A_2^{N+1}, A_2^{N+1}; 4, N+1, k] \\
&+ [A_{a+2}^{N+1}, A_{b+1}^{N+1}, A_2^{N+1}, A_2^{N+1}; 4, N+1, k]) \\
&- 3[A_{a+1}^{N+1}, A_{b+1}^{N+1}, A_3^{N+1}, A_2^{N+1}; 4, N+1, k] \\
&+ 6([A_{a+3}^{N+1}, A_{b+1}^{N+1}, A_2^{N+1}; 4, N+1, k] + [A_{a+2}^{N+1}, A_{b+2}^{N+1}, A_2^{N+1}; 4, N+1, k] \\
&+ [A_{a+1}^{N+1}, A_{b+3}^{N+1}, A_2^{N+1}; 4, N+1, k]) \\
&+ 3([A_{a+2}^{N+1}, A_{b+1}^{N+1}, A_3^{N+1}; 4, N+1, k] + [A_{a+1}^{N+1}, A_{b+2}^{N+1}, A_3^{N+1}; 4, N+1, k]) \\
&+ 2[A_{a+1}^{N+1}, A_{b+1}^{N+1}, A_4^{N+1}; 4, N+1, k]) \\
&- 6([A_{a+4}^{N+1}, A_{b+1}^{N+1}; 4, N+1, k] + [A_{a+3}^{N+1}, A_{b+2}^{N+1}; 4, N+1, k] \\
&+ [A_{a+2}^{N+1}, A_{b+3}^{N+1}; 4, N+1, k] + [A_{a+1}^{N+1}, A_{b+4}^{N+1}; 4, N+1, k]) \\
&\quad (a+b = N-3+4(N-k))
\end{aligned}
\tag{5.84}
$$

$$
\begin{aligned}
&\quad G[A_a^{N+1} \cap H, A_b^{N+1} \cap H, A_1^{N+1} \cap H, A_1^{N+1} \cap H, A_1^{N+1} \cap H, A_1^{N+1} \cap H; 5, N+1, k] \\
&= [A_{a+1}^{N+1}, A_{b+1}^{N+1}, A_2^{N+1}, A_2^{N+1}A_2^{N+1}, A_2^{N+1}; 5, N+1, k] \\
&- 4([A_{a+1}^{N+1}, A_{b+2}^{N+1}, A_2^{N+1}, A_2^{N+1}, A_2^{N+1}; 5, N+1, k] \\
&+ [A_{a+2}^{N+1}, A_{b+1}^{N+1}, A_2^{N+1}, A_2^{N+1}, A_2^{N+1}; 5, N+1, k]) \\
&- 6[A_{a+1}^{N+1}, A_{b+1}^{N+1}, A_3^{N+1}, A_2^{N+1}, A_2^{N+1}; 5, N+1, k] \\
&+ 12([A_{a+3}^{N+1}, A_{b+1}^{N+1}, A_2^{N+1}, A_2^{N+1}; 5, N+1, k] \\
&+ [A_{a+2}^{N+1}, A_{b+2}^{N+1}, A_2^{N+1}, A_2^{N+1}; 5, N+1, k] + [A_{a+1}^{N+1}, A_{b+3}^{N+1}, A_2^{N+1}, A_2^{N+1}; 5, N+1, k]) \\
&+ 12([A_{a+2}^{N+1}, A_{b+1}^{N+1}, A_3^{N+1}, A_2^{N+1}; 5, N+1, k] \\
&+ [A_{a+1}^{N+1}, A_{b+2}^{N+1}, A_3^{N+1}, A_2^{N+1}; 5, N+1, k]) \\
&+ 8[A_{a+1}^{N+1}, A_{b+1}^{N+1}, A_4^{N+1}, A_2^{N+1}; 5, N+1, k] \\
&+ 3[A_{a+1}^{N+1}, A_{b+1}^{N+1}, A_3^{N+1}, A_3^{N+1}; 5, N+1, k] \\
&- 24([A_{a+4}^{N+1}, A_{b+1}^{N+1}, A_2^{N+1}; 5, N+1, k] + [A_{a+3}^{N+1}, A_{b+2}^{N+1}, A_2^{N+1}; 5, N+1, k] \\
&+ [A_{a+2}^{N+1}, A_{b+3}^{N+1}, A_2^{N+1}; 5, N+1, k] + [A_{a+1}^{N+1}, A_{b+4}^{N+1}, A_2^{N+1}; 5, N+1, k]) \\
&- 12([A_{a+3}^{N+1}, A_{b+1}^{N+1}, A_3^{N+1}; 5, N+1, k] + [A_{a+2}^{N+1}, A_{b+2}^{N+1}, A_3^{N+1}; 5, N+1, k] \\
&+ [A_{a+1}^{N+1}, A_{b+3}^{N+1}, A_3^{N+1}; 5, N+1, k]) \\
&- 8([A_{a+2}^{N+1}, A_{b+1}^{N+1}, A_4^{N+1}; 5, N+1, k] + [A_{a+1}^{N+1}, A_{b+2}^{N+1}, A_4^{N+1}; 5, N+1, k]) \\
&- 6[A_{a+1}^{N+1}, A_{b+1}^{N+1}, A_5^{N+1}; 5, N+1, k]) \\
&+ 24([A_{a+5}^{N+1}, A_{b+1}^{N+1}; 5, N+1, k] + [A_{a+4}^{N+1}, A_{b+2}^{N+1}; 5, N+1, k] \\
&+ [A_{a+3}^{N+1}, A_{b+3}^{N+1}; 5, N+1, k] + [A_{a+2}^{N+1}, A_{b+4}^{N+1}; 5, N+1, k] \\
&+ [A_{a+1}^{N+1}, A_{b+4}^{N+1}; 5, N+1, k]) \\
&\quad (a+b = N-3+5(N-k)).
\end{aligned}
\tag{5.85}
$$

By combining (5.85) with (5.82), we can obtain specialization results for quartics.

$$
\begin{aligned}
&\quad 16L_n^{N,k,4} + R \\
&= \frac{3}{2}L_{n-3}^4 + \frac{13}{2}L_{n-2}^4 + \frac{13}{2}L_{n-1}^4 + \frac{3}{2}L_n^4
\end{aligned}
$$

$$
\begin{aligned}
&+2L^1_{n-2}L^3_{n-2+N-k}+2L^1_{n-1}L^3_{n-2+N-k}+6L^1_nL^3_{n-2+N-k}\\
&+8L^1_{n-1}L^3_{n-1+N-k}+12L^1_nL^3_{n-1+N-k}+6L^1_nL^3_{n+N-k}\\
&+3L^2_{n-2}L^2_{n-1+2(N-k)}+7L^2_{n-1}L^2_{n-1+2(N-k)}+6L^2_nL^2_{n-1+2(N-k)}\\
&+10L^2_{n-1}L^2_{n+2(N-k)}+7L^2_nL^2_{n+2(N-k)}+3L^2_nL^2_{n+1+2(N-k)}\\
&+6L^3_{n-2}L^1_{n+3(N-k)}+12L^3_{n-1}L^1_{n+3(N-k)}+6L^3_nL^1_{n+3(N-k)}\\
&+8L^3_{n-1}L^1_{n+1+3(N-k)}+2L^3_nL^1_{n+1+3(N-k)}+2L^3_nL^1_{n+2+3(N-k)}\\
&+4L^1_{n-1}L^1_{n-1+N-k}L^2_{n-1+2(N-k)}+9L^1_nL^1_{n-1+N-k}L^2_{n-1+2(N-k)}\\
&+10L^1_nL^1_{n+N-k}L^2_{n-1+2(N-k)}+12L^1_nL^1_{n+N-k}L^2_{n+2(N-k)}\\
&+8L^1_{n-1}L^2_{n-1+N-k}L^1_{n+3(N-k)}+14L^1_nL^2_{n-1+N-k}L^1_{n+3(N-k)}\\
&+14L^1_nL^2_{n+N-k}L^1_{n+3(N-k)}+8L^1_nL^2_{n+N-k}L^1_{n+1+3(N-k)}\\
&+12L^2_{n-1}L^1_{n+2(N-k)}L^1_{n+3(N-k)}+10L^2_nL^1_{n+2(N-k)}L^1_{n+3(N-k)}\\
&+9L^2_nL^1_{n+1+2(N-k)}L^1_{n+3(N-k)}+4L^2_nL^1_{n+1+2(N-k)}L^1_{n+1+3(N-k)}\\
&+16L^1_nL^1_{n+N-k}L^1_{n+2(N-k)}L^1_{n+3(N-k)} \qquad (5.86)
\end{aligned}
$$

In (5.86), we omit $N+1,k$ from $L_n^{N+1,k,d}$ for brevity. We will omit them from now on. Next, we determine the contribution from connected reducible curves R indirectly. Assuming that this specialization result exhausts all the terms that appear in "true " recursion relations (this is true for $d \leq 3$ rational curves), we set unknown coefficients for terms of reducible curves considering symmetry of coefficients, which can be seen in the specialization results.

$$
\begin{aligned}
&L_n^{N,k,4}\\
&=\frac{3}{32}L^4_{n-3}+\frac{13}{32}L^4_{n-2}+\frac{13}{32}L^4_{n-1}+\frac{3}{32}L^4_n\\
&+a_1L^1_{n-2}L^3_{n-2+N-k}+a_2L^1_{n-1}L^3_{n-2+N-k}+a_3L^1_nL^3_{n-2+N-k}\\
&+a_4L^1_{n-1}L^3_{n-1+N-k}+a_5L^1_nL^3_{n-1+N-k}+a_6L^1_nL^3_{n+N-k}\\
&+b_1L^2_{n-2}L^2_{n-1+2(N-k)}+b_2L^2_{n-1}L^2_{n-1+2(N-k)}+b_3L^2_nL^2_{n-1+2(N-k)}\\
&+b_4L^2_{n-1}L^2_{n+2(N-k)}+b_2L^2_nL^2_{n+2(N-k)}+b_1L^2_nL^2_{n+1+2(N-k)}\\
&+a_6L^3_{n-2}L^1_{n+3(N-k)}+a_5L^3_{n-1}L^1_{n+3(N-k)}+a_3L^3_nL^1_{n+3(N-k)}\\
&+a_4L^3_{n-1}L^1_{n+1+3(N-k)}+a_2L^3_nL^1_{n+1+3(N-k)}+a_1L^3_nL^1_{n+2+3(N-k)}\\
&+c_1L^1_{n-1}L^1_{n-1+(N-k)}L^2_{n-1+2(N-k)}+c_2L^1_nL^1_{n-1+N-k}L^2_{n-1+2(N-k)}\\
&+c_3L^1_nL^1_{n+N-k}L^2_{n-1+2(N-k)}+c_4L^1_nL^1_{n+N-k}L^2_{n+2(N-k)}\\
&+d_1L^1_{n-1}L^2_{n-1+N-k}L^1_{n+3(N-k)}+d_2L^1_nL^2_{n-1+N-k}L^1_{n+3(N-k)}\\
&+d_2L^1_nL^2_{n+N-k}L^1_{n+3(N-k)}+d_1L^1_nL^2_{n+N-k}L^1_{n+1+3(N-k)}\\
&+c_4L^2_{n-1}L^1_{n+2(N-k)}L^1_{n+3(N-k)}+c_3L^2_nL^1_{n+2(N-k)}L^1_{n+3(N-k)}\\
&+c_2L^2_nL^1_{n+1+2(N-k)}L^1_{n+3(N-k)}+c_1L^2_nL^1_{n+1+2(N-k)}L^1_{n+1+3(N-k)}\\
&+e_1L^1_nL^1_{n+N-k}L^1_{n+2(N-k)}L^1_{n+3(N-k)} \qquad (5.87)
\end{aligned}
$$

From conjectural characteristics of recursion relation that imposes $\gamma_0^{N,k,4}=\gamma_0^{N+1,k,4}$, we obtain some constraints on these unknown coefficients.

$$
\begin{aligned}
&a_3=\frac{2}{9},\;\; a_2+a_5=\frac{7}{9},\;\; a_1+a_4+a_6=1\\
&b_3=\frac{1}{4},\;\; 2b_2=\frac{3}{4},\;\; 2b_1+b_4=1
\end{aligned}
$$

$$
\begin{aligned}
&c_2 = \frac{1}{3}, \quad c_3 = \frac{1}{2}, \quad c_1 + c_4 = 1 \\
&2d_1 = 1, \quad d_2 = d_3 = \frac{2}{3} \\
&e_1 = 1
\end{aligned} \tag{5.88}
$$

If we compare (5.86) with (5.88), we can see $a_1+a_4+a_6 = 1, 2b_1+b_4 = 1, , c_1+c_4 = 1, 2d_1 = 1, e_1 = 1$ are automatically satisfied in (5.86). And we heuristically set $a_1 = \frac{1}{8}, a_4 = \frac{1}{2}, a_6 = \frac{3}{8}, b_1 = \frac{3}{16}, b_4 = \frac{5}{8}, c_1 = \frac{1}{4}, c_4 = \frac{3}{4}, d_1 = \frac{1}{2}, e_1 = 1$. We have to note some combinatorial relation on these coefficients that can be seen from (5.86).

$$
\begin{aligned}
&\frac{3}{32}x^3 + \frac{13}{32}x^2y + \frac{13}{32}xy^2 + \frac{3}{32}y^3 = (\frac{3x+y}{4})(\frac{2x+2y}{4})(\frac{x+3y}{4}) \\
&\frac{1}{8}x^2 + \frac{1}{2}xy + \frac{3}{8}y^2 = (\frac{2x+2y}{4})(\frac{x+3y}{4}), \quad \frac{3}{16}x^2 + \frac{5}{8}xy + \frac{3}{16}y^2 = (\frac{3x+y}{4})(\frac{x+3y}{4}) \\
&\frac{3}{8}x^2 + \frac{1}{2}xy + \frac{1}{8}y^2 = (\frac{3x+y}{4})(\frac{2x+2y}{4}) \\
&\frac{1}{4}x + \frac{3}{4}y = (\frac{x+3y}{4}), \quad \frac{1}{2}x + \frac{1}{2}y = (\frac{2x+2y}{4}) \\
&\frac{3}{4}x + \frac{1}{4}y = (\frac{3x+y}{4})
\end{aligned} \tag{5.89}
$$

As a summary of discussion given so far, we propose the following.

Conjecture 4 *The prototype result obtained from specialization exhausts all the addends that appear in the "true" recursive formula and coefficients described by the following generating polynomial remain unchanged after subtraction of contribution from "R" term.*

$$
\prod_{j=1}^{d-1}(\frac{jx+(d-j)y}{d} + z_j) \tag{5.90}
$$

Examples

$$
\begin{aligned}
d = 2 \qquad & (\frac{x+y}{2}) + z_1 \\
d = 3 \qquad & (\frac{2x^2+5xy+2y^2}{9}) + (\frac{2x+y}{3})z_1 + (\frac{x+2y}{3})z_2 + z_1z_2 \\
d = 4 \qquad & (\frac{3x^3+13x^2y+13xy^2+3y^3}{32}) \\
& +(\frac{x^2+4xy+3y^2}{8})z_3 + (\frac{3x^2+10xy+3y^2}{16})z_2 + (\frac{3x^2+4xy+y^2}{8})z_1 \\
& +(\frac{3x+y}{4})z_1z_2 + (\frac{x+y}{2})z_1z_3 + (\frac{x+3y}{4})z_2z_3 \\
& +z_1z_2z_3 \\
d = 5 \qquad & (\frac{24x^4+154x^3y+269x^2y^2+154xy^3+24y^4}{625}) \\
& +(\frac{6x^3+37x^2y+58xy^2+24y^3}{125})z_4 + (\frac{8x^3+46x^2y+59xy^2+12y^3}{125})z_3 \\
& +(\frac{12x^3+59x^2y+46xy^2+8y^3}{125})z_2 + (\frac{24x^3+58x^2y+37xy^2+6y^3}{125})z_1 \\
& +(\frac{2x^2+11xy+12y^2}{25})z_3z_4 + (\frac{6x^2+13xy+6y^2}{25})z_1z_4 + (\frac{12x^2+11xy+2y^2}{25})z_1z_2
\end{aligned}
$$

$$+(\frac{3x^2+14xy+8y^2}{25})z_2z_4+(\frac{4x^2+17xy+4y^2}{25})z_2z_3+(\frac{8x^2+14xy+3y^2}{25})z_1z_3$$
$$+(\frac{4x+y}{5})z_1z_2z_3+(\frac{3x+2y}{5})z_1z_2z_4+(\frac{2x+3y}{5})z_1z_3z_4+(\frac{x+4y}{5})z_2z_3z_4$$
$$+z_1z_2z_3z_4 \tag{5.91}$$

Then we go back to the argument of quartic curves. Remaining unknown coefficient is only a_2 (a_5). Then we use some numerical results obtained from torus action method.

$$\begin{aligned}
&H^*_{q,e}(M_9^7) \\
\mathcal{O}_e\cdot\mathcal{O}_e &= \mathcal{O}_{e^2}+5040q \\
\mathcal{O}_e\cdot\mathcal{O}_{e^2} &= \mathcal{O}_{e^3}+56196\mathcal{O}_eq \\
\mathcal{O}_e\cdot\mathcal{O}_{e^3} &= \mathcal{O}_{e^4}+200452\mathcal{O}_{e^2}q+2056259520q^2 \\
\mathcal{O}_e\cdot\mathcal{O}_{e^4} &= \mathcal{O}_{e^5}+300167\mathcal{O}_{e^3}q+24699506832\mathcal{O}_eq^2 \\
\mathcal{O}_e\cdot\mathcal{O}_{e^5} &= \mathcal{O}_{e^6}+200452\mathcal{O}_{e^4}q+537516856624\mathcal{O}_{e^2}q^2 \\
&\quad +534155202302400q^3 \\
\mathcal{O}_e\cdot\mathcal{O}_{e^6} &= \mathcal{O}_{e^7}+56196\mathcal{O}_{e^5}q+24699506832\mathcal{O}_{e^3}q^2 \\
&\quad +1920365635990032\mathcal{O}_eq^3 \\
\mathcal{O}_e\cdot\mathcal{O}_{e^7} &= 5040\mathcal{O}_{e^6}q+2056259520\mathcal{O}_{e^4}q^2 \\
&\quad +534155202302400\mathcal{O}_{e^2}q^3 \\
&\quad +5112982794486067200q^4
\end{aligned} \tag{5.92}$$

$$\begin{aligned}
&H^*_{q,e}(M_8^7) \\
\mathcal{O}_e\cdot\mathcal{O}_e &= \mathcal{O}_{e^2}+51156\mathcal{O}_eq+1311357600q^2 \\
\mathcal{O}_e\cdot\mathcal{O}_{e^2} &= \mathcal{O}_{e^3}+195412\mathcal{O}_{e^2}q+24642483768\mathcal{O}_eq^2+675477943761600q^3 \\
\mathcal{O}_e\cdot\mathcal{O}_{e^3} &= \mathcal{O}_{e^4}+295127\mathcal{O}_{e^3}q+99394671712\mathcal{O}_{e^2}q^2+126228416888463120\mathcal{O}_eq^3 \\
&\quad +3528264665849188608000q^4 \\
\mathcal{O}_e\cdot\mathcal{O}_{e^4} &= \mathcal{O}_{e^5}+195412\mathcal{O}_{e^4}q+99394671712\mathcal{O}_{e^3}q^2+32755090390395744\mathcal{O}_{e^2}q^3 \\
&\quad +4092145211387781662688\mathcal{O}_eq^4+\cdots \\
\mathcal{O}_e\cdot\mathcal{O}_{e^5} &= \mathcal{O}_{e^6}+51156\mathcal{O}_{e^5}q+24642483768\mathcal{O}_{e^4}q^2+126228416888463120\mathcal{O}_{e^3}q^3 \\
&\quad +4092145211387781662688\mathcal{O}_{e^2}q^4+\cdots \\
\mathcal{O}_e\cdot\mathcal{O}_{e^6} &= 1311357600\mathcal{O}_{e^5}q^2+675477943761600\mathcal{O}_{e^4}q^3 \\
&\quad +3528264665849188608000\mathcal{O}_{e^3}q^4+\cdots
\end{aligned} \tag{5.93}$$

Then applying the recursion relation with unknown a_2 for 3528264665849188608000, we obtain the following equation.

$$\frac{109466}{3}=(\frac{7}{9}-a_2)\cdot 5040+a_2\cdot 200452 \tag{5.94}$$

And we have $a_2=\frac{1}{6}$. The final result is,

$$\begin{aligned}
L_n^{N,k,4} &= \frac{1}{32}(3L^4_{n-3}+13L^4_{n-2}+13L^4_{n-1}+3L^4_n) \\
&\quad +\frac{1}{72}(9L^1_{n-2}L^3_{n-2+N-k}+12L^1_{n-1}L^3_{n-2+N-k}+16L^1_nL^3_{n-2+N-k}
\end{aligned}$$

$$
\begin{aligned}
&+36L^1_{n-1}L^3_{n-1+N-k}+44L^1_nL^3_{n-1+N-k}+27L^1_nL^3_{n+N-k})\\
&+\frac{1}{16}(3L^2_{n-2}L^2_{n-1+2(N-k)}+6L^2_{n-1}L^2_{n-1+2(N-k)}+4L^2_nL^2_{n-1+2(N-k)}\\
&+10L^2_{n-1}L^2_{n+2(N-k)}+6L^2_nL^2_{n+2(N-k)}+3L^2_nL^2_{n+1+2(N-k)})\\
&+\frac{1}{72}(27L^3_{n-2}L^1_{n+3(N-k)}+44L^3_{n-1}L^1_{n+3(N-k)}+16L^3_nL^1_{n+3(N-k)}\\
&+36L^3_{n-1}L^1_{n+1+3(N-k)}+12L^3_nL^1_{n+1+3(N-k)}+9L^3_nL^1_{n+2+3(N-k)})\\
&+\frac{1}{12}(3L^1_{n-1}L^1_{n-1+N-k}L^2_{n-1+2(N-k)}+4L^1_nL^1_{n-1+N-k}L^2_{n-1+2(N-k)}\\
&+6L^1_nL^1_{n+N-k}L^2_{n-1+2(N-k)}+9L^1_nL^1_{n+N-k}L^2_{n+2(N-k)})\\
&+\frac{1}{6}(3L^1_{n-1}L^2_{n-1+N-k}L^1_{n+3(N-k)}+4L^1_nL^2_{n-1+N-k}L^1_{n+3(N-k)}\\
&+4L^1_nL^2_{n+N-k}L^1_{n+3(N-k)}+3L^1_nL^2_{n+N-k}L^1_{n+1+3(N-k)})\\
&+\frac{1}{12}(9L^2_{n-1}L^1_{n+2(N-k)}L^1_{n+3(N-k)}+6L^2_nL^1_{n+2(N-k)}L^1_{n+3(N-k)}\\
&+4L^2_nL^1_{n+1+2(N-k)}L^1_{n+3(N-k)}+3L^2_nL^1_{n+1+2(N-k)}L^1_{n+1+3(N-k)})\\
&+L^1_nL^1_{n+N-k}L^1_{n+2(N-k)}L^1_{n+3(N-k)}. \qquad (5.95)
\end{aligned}
$$

Of course, we have the following equality.

$$
\begin{aligned}
& (13/32)\cdot 5112982794486067200\\
+\;& (1/6)\cdot 5040\cdot 534155202302400+(2/9)\cdot 56196\cdot 534155202302400\\
+\;& (1/2)\cdot 5040\cdot 1920365635990032+(11/18)\cdot 56196\cdot 1920365635990032\\
+\;& (3/8)\cdot 56196\cdot 534155202302400+(3/8)\cdot 2056259520\cdot 53751685624\\
+\;& (1/4)\cdot 24699506832\cdot 53751685624+(5/8)\cdot 2056259520\cdot 24699506832\\
+\;& (3/8)\cdot 24699506832\cdot 24699506832+(3/16)\cdot 24699506832\cdot 2056259520\\
+\;& (11/18)\cdot 534155202302400\cdot 200452+(2/9)\cdot 1920365635990032\cdot 200452\\
+\;& (1/2)\cdot 534155202302400\cdot 56196+(1/6)\cdot 1920365635990032\cdot 56196\\
+\;& (1/8)\cdot 1920365635990032\cdot 5040+(1/4)\cdot 5040\cdot 56196\cdot 53751685624\\
+\;& (1/3)\cdot 56196\cdot 56196\cdot 53751685624+(1/2)\cdot 56196\cdot 200452\cdot 53751685624\\
+\;& (3/4)\cdot 56196\cdot 200452\cdot 24699506832+(1/2)\cdot 5040\cdot 24699506832\cdot 200452\\
+\;& (2/3)\cdot 56196\cdot 24699506832\cdot 200452+(2/3)\cdot 56196\cdot 53751685624\cdot 200452\\
+\;& (1/2)\cdot 56196\cdot 53751685624\cdot 56196+(3/4)\cdot 2056259520\cdot 300167\cdot 200452\\
+\;& (1/2)\cdot 24699506832\cdot 300167\cdot 200452+(1/3)\cdot 24699506832\cdot 200452\cdot 200452\\
+\;& (1/4)\cdot 24699506832\cdot 200452\cdot 56196+56196\cdot 200452\cdot 300167\cdot 200452\\
=\;& 4092145211387781662688 \qquad (5.96)
\end{aligned}
$$

We further checked numerically the previous conjecture of virtual quantum cohomology ring of Calabi-Yau hypersurfaces in CP^{k-1}.

$$
\begin{aligned}
k^{4k}a_4 &= \tilde{L}^{k,k,4}_0\\
k^{4k}b_4 &= \frac{1}{4}\tilde{L}^{k,k,4}_1+\frac{1}{3}\tilde{L}^{k,k,3}_1\cdot\tilde{L}^{k,k,1}_0+\frac{1}{2}\tilde{L}^{k,k,2}_1\cdot\tilde{L}^{k,k,2}_0+\tilde{L}^{k,k,1}_1\cdot\tilde{L}^{k,k,3}_0. \qquad (5.97)
\end{aligned}
$$

Moreover, we obtained correct Gromov-Witten invariant for M^5_5 case under the assumption of (4.66). To write out general recursive formula is tedious, and we give the result for M^5_5 in the following.

$$(c_4-b_4)-c_3\cdot b_1+2\cdot b_1\cdot b_3-b_3\cdot c_1-c_2\cdot b_2$$

$$
\begin{aligned}
&+c_2\cdot(b_1)^2-3\cdot b_2\cdot(b_1)^2+2\cdot b_2\cdot b_1\cdot c_1-c_1\cdot(b_1)^3+(b_1)^4+(b_2)^2\\
&+(c_1-b_1)\cdot(-(1/3)\cdot b_3+(3/2)\cdot b_1\cdot b_2-(3/2)\cdot(b_1)^3)\\
&+2\cdot(c_2-b_2-c_1\cdot b_1+(b_1)^2)\cdot(-(1/2)\cdot b_2+(b_1)^2)\\
&+3\cdot(c_3-b_3-c_2\cdot b_1+2\cdot b_2\cdot b_1-c_1\cdot b_2+c_1\cdot(b_1)^2-(b_1)^3)\cdot(-b_1)\\
&+(c_1-b_1)\cdot(-b_1)\cdot(-(1/2)\cdot b_2+(b_1)^2)+2\cdot(c_2-b_2-c_1\cdot b_1+(b_1)^2)\cdot((b_1)^2)\\
&-(1/6)\cdot(c_1-b_1)\cdot((b_1)^3)\\
=\ &3103585359375\\
&\text{where}\\
&b_1=770,\ b_2=1435650,\ b_3=3225308000,\ b_4=7894629141250\\
&c_1=1345,\ c_2=3296525,\ c_3=8940963625,\ c_4=25306794813125
\end{aligned}
$$

Remark 1

Instead of using numerical results, we can derive $a_2=\frac{1}{6}$ from information of coefficients of hypergeometric series. We will explain it in determination of recursive formula for quintic curves.

Remark 2

We can formally generalize the polynomial description of recursive formulas (5.91) to include all the terms that appear in recursive formulas. For example,

$$
\begin{aligned}
d=3 \qquad & (\frac{2x^2+5xy+2y^2}{9})+(\frac{2x+y}{3}+\frac{1}{2}z_1)z_1+(\frac{x+2y}{3}+\frac{1}{2}z_2)z_2+z_1z_2\\
d=4 \qquad & (\frac{3x^3+13x^2y+13xy^2+3y^3}{32})\\
&+(\frac{x^2+4xy+3y^2}{8}+(\frac{3x+11y}{18})z_3+\frac{2}{9}(z_3)^2)z_3\\
&+(\frac{3x^2+10xy+3y^2}{16}+(\frac{3x+3y}{8})z_2+\frac{1}{4}(z_2)^2)z_2\\
&+(\frac{3x^2+4xy+y^2}{8}+(\frac{11x+3y}{18})z_1+\frac{2}{9}(z_1)^2)z_1\\
&+(\frac{3x+y}{4}+\frac{1}{2}z_1+\frac{1}{3}z_2)z_1z_2+(\frac{x+y}{2}+\frac{2}{3}z_1+\frac{2}{3}z_3)z_1z_3\\
&+(\frac{x+3y}{4}+\frac{1}{3}z_2+\frac{1}{2}z_3)z_2z_3\\
&+z_1z_2z_3
\end{aligned} \tag{5.98}
$$

From this formula, we can speculate that the number of addends in recursive formula for degree d curves is equal to the number of degree $d-1$ monomials of $d+1$ variables, i.e., ${}_{2d-1}C_d$. But this generalization does not have simple factorization property like (5.91). □

Then we go on to determination of recursive formula for quintic curves. The prototype result from specialization approach is the following.

$$
\begin{aligned}
&125L_n^{N,k,5}+R\\
=\ &\frac{24}{5}L^5_{n-4}+\frac{154}{5}L^5_{n-3}+\frac{269}{5}L^5_{n-2}+\frac{154}{5}L^5_{n-1}+\frac{24}{5}L^5_n\\
&+6L^1_{n-3}L^4_{n-3+N-k}+9L^1_{n-2}L^4_{n-3+N-k}+2L^1_{n-1}L^4_{n-3+N-k}\\
&+24L^1_nL^4_{n-3+N-k}+37L^1_{n-2}L^4_{n-2+N-k}+38L^1_{n-1}L^4_{n-2+N-k}\\
&+80L^1_nL^4_{n-2+N-k}+58L^1_{n-1}L^4_{n-1+N-k}+72L^1_nL^4_{n-1+N-k}\\
&+24L^1_nL^4_{n+N-k}\\
&+8L^2_{n-3}L^3_{n-2+2(N-k)}+12L^2_{n-2}L^3_{n-2+2(N-k)}+28L^2_{n-1}L^3_{n-2+2(N-k)}\\
&+24L^2_nL^3_{n-2+2(N-k)}+46L^2_{n-2}L^3_{n-1+2(N-k)}+74L^2_{n-1}L^3_{n-1+2(N-k)}
\end{aligned}
$$

$$+54L_n^2L_{n-1+2(N-k)}^3+59L_{n-1}^2L_{n+2(N-k)}^3+33L_n^2L_{n+2(N-k)}^3$$
$$+12L_n^2L_{n+1+2(N-k)}^3$$
$$+12L_{n-3}^3L_{n-1+3(N-k)}^2+33L_{n-2}^3L_{n-1+3(N-k)}^2+54L_{n-1}^3L_{n-1+3(N-k)}^2$$
$$+24L_n^3L_{n-1+3(N-k)}^2+59L_{n-2}^3L_{n+3(N-k)}^2+74L_{n-1}^3L_{n+3(N-k)}^2$$
$$+28L_n^3L_{n+3(N-k)}^2+46L_{n-1}^3L_{n+1+3(N-k)}^2+12L_n^3L_{n+1+3(N-k)}^2$$
$$+8L_n^3L_{n+2+3(N-k)}^2$$
$$+24L_{n-3}^4L_{n+4(N-k)}^1+72L_{n-2}^4L_{n+4(N-k)}^1+80L_{n-1}^4L_{n+4(N-k)}^1$$
$$+24L_n^4L_{n+4(N-k)}^1+58L_{n-2}^4L_{n+1+4(N-k)}^1+38L_{n-1}^4L_{n+1+4(N-k)}^1$$
$$+2L_n^4L_{n+1+4(N-k)}^1+37L_{n-1}^4L_{n+2+4(N-k)}^1+9L_n^4L_{n+2+4(N-k)}^1$$
$$+6L_n^4L_{n+3+4(N-k)}^1$$
$$+10L_{n-2}^1L_{n-2+N-k}^1L_{n-2+2(N-k)}^3+3L_{n-1}^1L_{n-2+N-k}^1L_{n-2+2(N-k)}^3$$
$$+32L_n^1L_{n-2+N-k}^1L_{n-2+2(N-k)}^3+14L_{n-1}^1L_{n-1+N-k}^1L_{n-2+2(N-k)}^3$$
$$+43L_n^1L_{n-1+N-k}^1L_{n-2+2(N-k)}^3+42L_n^1L_{n+N-k}^1L_{n-2+2(N-k)}^3$$
$$+55L_{n-1}^1L_{n-1+N-k}^1L_{n-1+2(N-k)}^3+96L_n^1L_{n-1+N-k}^1L_{n-1+2(N-k)}^3$$
$$+93L_n^1L_{n+N-k}^1L_{n-1+2(N-k)}^3+60L_n^1L_{n+N-k}^1L_{n+2(N-k)}^3$$
$$+30L_{n-2}^1L_{n-2+N-k}^3L_{n+4(N-k)}^1+29L_{n-1}^1L_{n-2+N-k}^3L_{n+4(N-k)}^1$$
$$+78L_n^1L_{n-2+N-k}^3L_{n+4(N-k)}^1+86L_{n-1}^1L_{n-1+N-k}^3L_{n+4(N-k)}^1$$
$$+123L_n^1L_{n-1+N-k}^3L_{n+4(N-k)}^1+78L_n^1L_{n+N-k}^3L_{n+4(N-k)}^1$$
$$+65L_{n-1}^1L_{n-1+N-k}^3L_{n+1+4(N-k)}^1+86L_n^1L_{n-1+N-k}^3L_{n+1+4(N-k)}^1$$
$$+29L_n^1L_{n+N-k}^3L_{n+1+4(N-k)}^1+30L_n^1L_{n+N-k}^3L_{n+2+4(N-k)}^1$$
$$+60L_{n-2}^3L_{n+3(N-k)}^1L_{n+4(N-k)}^1+93L_{n-1}^3L_{n+3(N-k)}^1L_{n+4(N-k)}^1$$
$$+42L_n^3L_{n+3(N-k)}^1L_{n+4(N-k)}^1+96L_{n-1}^3L_{n+1+3(N-k)}^1L_{n+4(N-k)}^1$$
$$+43L_n^3L_{n+1+3(N-k)}^1L_{n+4(N-k)}^1+32L_n^3L_{n+2+3(N-k)}^1L_{n+4(N-k)}^1$$
$$+55L_{n-1}^3L_{n+1+3(N-k)}^1L_{n+1+4(N-k)}^1+14L_n^3L_{n+1+3(N-k)}^1L_{n+1+4(N-k)}^1$$
$$+3L_n^3L_{n+2+3(N-k)}^1L_{n+1+4(N-k)}^1+10L_n^3L_{n+2+3(N-k)}^1L_{n+2+4(N-k)}^1$$
$$+15L_{n-2}^1L_{n-2+N-k}^2L_{n-1+3(N-k)}^2+10L_{n-1}^1L_{n-2+N-k}^2L_{n-1+3(N-k)}^2$$
$$+48L_n^1L_{n-2+N-k}^2L_{n-1+3(N-k)}^2+42L_{n-1}^1L_{n-1+N-k}^2L_{n-1+3(N-k)}^2$$
$$+76L_n^1L_{n-1+N-k}^2L_{n-1+3(N-k)}^2+60L_n^1L_{n+N-k}^2L_{n-1+3(N-k)}^2$$
$$+70L_{n-1}^1L_{n-1+N-k}^2L_{n+3(N-k)}^2+104L_n^1L_{n-1+N-k}^2L_{n+3(N-k)}^2$$
$$+76L_n^1L_{n+N-k}^2L_{n+3(N-k)}^2+40L_n^1L_{n+N-k}^2L_{n+1+3(N-k)}^2$$
$$+20L_{n-2}^2L_{n-1+2(N-k)}^1L_{n-1+3(N-k)}^2+44L_{n-1}^2L_{n-1+2(N-k)}^1L_{n-1+3(N-k)}^2$$
$$+36L_n^2L_{n-1+2(N-k)}^1L_{n-1+3(N-k)}^2+59L_{n-1}^2L_{n+2(N-k)}^1L_{n-1+3(N-k)}^2$$
$$+45L_n^2L_{n+2(N-k)}^1L_{n-1+3(N-k)}^2+36L_n^2L_{n+1+2(N-k)}^1L_{n-1+3(N-k)}^2$$
$$+85L_{n-1}^2L_{n+2(N-k)}^1L_{n+3(N-k)}^2+59L_n^2L_{n+2(N-k)}^1L_{n+3(N-k)}^2$$
$$+44L_n^2L_{n+1+2(N-k)}^1L_{n+3(N-k)}^2+20L_n^2L_{n+1+2(N-k)}^1L_{n+1+3(N-k)}^2$$
$$+40L_{n-2}^2L_{n-1+2(N-k)}^2L_{n+4(N-k)}^1+76L_{n-1}^2L_{n-1+2(N-k)}^2L_{n+4(N-k)}^1$$

$$
\begin{aligned}
&+60L_n^2L_{n-1+2(N-k)}^2L_{n+4(N-k)}^1+104L_{n-1}^2L_{n+2(N-k)}^2L_{n+4(N-k)}^1\\
&+76L_n^2L_{n+2(N-k)}^2L_{n+4(N-k)}^1+48L_n^2L_{n+1+2(N-k)}^2L_{n+4(N-k)}^1\\
&+70L_{n-1}^2L_{n+2(N-k)}^2L_{n+1+4(N-k)}^1+42L_n^2L_{n+2(N-k)}^2L_{n+1+4(N-k)}^1\\
&+10L_n^2L_{n+1+2(N-k)}^2L_{n+1+4(N-k)}^1+15L_n^2L_{n+1+2(N-k)}^2L_{n+2+4(N-k)}^1\\
&+25L_{n-1}^1L_{n-1+N-k}^1L_{n-1+2(N-k)}^1L_{n-1+3(N-k)}^2+64L_n^1L_{n-1+N-k}^1L_{n-1+2(N-k)}^1L_{n-1+3(N-k)}^2\\
&+63L_n^1L_{n+N-k}^1L_{n-1+2(N-k)}^1L_{n-1+3(N-k)}^2+76L_n^1L_{n+N-k}^1L_{n+2(N-k)}^1L_{n-1+3(N-k)}^2\\
&+100L_n^1L_{n+N-k}^1L_{n+2(N-k)}^1L_{n+3(N-k)}^2+50L_{n-1}^1L_{n-1+N-k}^1L_{n-1+2(N-k)}^2L_{n+4(N-k)}^1\\
&+101L_n^1L_{n-1+N-k}^1L_{n-1+2(N-k)}^2L_{n+4(N-k)}^1+102L_n^1L_{n+N-k}^1L_{n-1+2(N-k)}^2L_{n+4(N-k)}^1\\
&+122L_n^1L_{n+N-k}^1L_{n+2(N-k)}^2L_{n+4(N-k)}^1+75L_n^1L_{n+N-k}^1L_{n+2(N-k)}^2L_{n+1+4(N-k)}^1\\
&+75L_{n-1}^1L_{n-1+N-k}^2L_{n+3(N-k)}^1L_{n+4(N-k)}^1+122L_n^1L_{n-1+N-k}^2L_{n+3(N-k)}^1L_{n+4(N-k)}^1\\
&+102L_n^1L_{n+N-k}^2L_{n+3(N-k)}^1L_{n+4(N-k)}^1+101L_n^1L_{n+N-k}^2L_{n+1+3(N-k)}^1L_{n+4(N-k)}^1\\
&+50L_n^1L_{n+N-k}^2L_{n+1+3(N-k)}^1L_{n+1+4(N-k)}^1+100L_{n-1}^2L_{n+2(N-k)}^1L_{n+3(N-k)}^1L_{n+4(N-k)}^1\\
&+76L_n^2L_{n+2(N-k)}^1L_{n+3(N-k)}^1L_{n+4(N-k)}^1+63L_n^2L_{n+1+2(N-k)}^1L_{n+3(N-k)}^1L_{n+4(N-k)}^1\\
&+64L_n^2L_{n+1+2(N-k)}^1L_{n+1+3(N-k)}^1L_{n+4(N-k)}^1+25L_n^2L_{n+1+2(N-k)}^1L_{n+1+3(N-k)}^1L_{n+1+4(N-k)}^1\\
&+125L_n^1L_{n+N-k}^1L_{n+2(N-k)}^1L_{n+3(N-k)}^1L_{n+4(N-k)}^1 \qquad (5.99)
\end{aligned}
$$

We can see combinatorial characteristics of coefficients described in (5.91) also in this case and we assume these coefficients are true. Then we determined the remaining unknown coefficients of true recursive formula using the following method.

We first obtain some linear relations between them using Conjecture 2. Then, now that we have obtained recursive formulas for $d \leq 4$ curves and assume Conjecture 4, successive application of recursive formula from $N \geq 2k$ region (in this region, what we need is only the Schubert numbers !) to $N = k$ region results in linear function of the remaining unknown coefficients. Then from information of coefficients of hypergeometric series a_d and b_d in Conjecture 3, we can obtain infinite number of linear relations on them, varying N. The final result we obtained is the following.

$$
\begin{aligned}
&L_n^{N,k,5}\\
=\ &\frac{24}{625}L_{n-4}^5+\frac{154}{625}L_{n-3}^5+\frac{269}{625}L_{n-2}^5+\frac{154}{625}L_{n-1}^5+\frac{24}{625}L_n^5\\
&+\frac{6}{125}L_{n-3}^1L_{n-3+N-k}^4+\frac{3}{50}L_{n-2}^1L_{n-3+N-k}^4+\frac{3}{40}L_{n-1}^1L_{n-3+N-k}^4\\
&+\frac{3}{32}L_n^1L_{n-3+N-k}^4+\frac{37}{125}L_{n-2}^1L_{n-2+N-k}^4+\frac{71}{200}L_{n-1}^1L_{n-2+N-k}^4\\
&+\frac{17}{40}L_n^1L_{n-2+N-k}^4+\frac{58}{125}L_{n-1}^1L_{n-1+N-k}^4+\frac{393}{800}L_n^1L_{n-1+N-k}^4\\
&+\frac{24}{125}L_n^1L_{n+N-k}^4\\
&+\frac{8}{125}L_{n-3}^2L_{n-2+2(N-k)}^3+\frac{8}{75}L_{n-2}^2L_{n-2+2(N-k)}^3+\frac{8}{45}L_{n-1}^2L_{n-2+2(N-k)}^3\\
&+\frac{1}{9}L_n^2L_{n-2+2(N-k)}^3+\frac{46}{125}L_{n-2}^2L_{n-1+2(N-k)}^3+\frac{122}{225}L_{n-1}^2L_{n-1+2(N-k)}^3\\
&+\frac{29}{90}L_n^2L_{n-1+2(N-k)}^3+\frac{59}{125}L_{n-1}^2L_{n+2(N-k)}^3+\frac{6}{25}L_n^2L_{n+2(N-k)}^3\\
&+\frac{12}{125}L_n^2L_{n+1+2(N-k)}^3\\
&+\frac{12}{125}L_{n-3}^3L_{n-1+3(N-k)}^2+\frac{6}{25}L_{n-2}^3L_{n-1+3(N-k)}^2+\frac{29}{90}L_{n-1}^3L_{n-1+3(N-k)}^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{9}L_n^3L_{n-1+3(N-k)}^2+\frac{59}{125}L_{n-2}^3L_{n+3(N-k)}^2+\frac{122}{225}L_{n-1}^3L_{n+3(N-k)}^2\\
&+\frac{8}{45}L_n^3L_{n+3(N-k)}^2+\frac{46}{125}L_{n-1}^3L_{n+1+3(N-k)}^2+\frac{8}{75}L_n^3L_{n+1+3(N-k)}^2\\
&+\frac{8}{125}L_n^3L_{n+2+3(N-k)}^2\\
&+\frac{24}{125}L_{n-3}^4L_{n+4(N-k)}^1+\frac{393}{800}L_{n-2}^4L_{n+4(N-k)}^1+\frac{17}{40}L_{n-1}^4L_{n+4(N-k)}^1\\
&+\frac{3}{32}L_n^4L_{n+4(N-k)}^1+\frac{58}{125}L_{n-2}^4L_{n+1+4(N-k)}^1+\frac{71}{200}L_{n-1}^4L_{n+1+4(N-k)}^1\\
&+\frac{3}{40}L_n^4L_{n+1+4(N-k)}^1+\frac{37}{125}L_{n-1}^4L_{n+2+4(N-k)}^1+\frac{3}{50}L_n^4L_{n+2+4(N-k)}^1\\
&+\frac{6}{125}L_n^4L_{n+3+4(N-k)}^1\\
&+\frac{2}{25}L_{n-2}^1L_{n-2+N-k}^1L_{n-2+2(N-k)}^3+\frac{1}{10}L_{n-1}^1L_{n-2+N-k}^1L_{n-2+2(N-k)}^3\\
&+\frac{1}{8}L_n^1L_{n-2+N-k}^1L_{n-2+2(N-k)}^3+\frac{2}{15}L_{n-1}^1L_{n-1+N-k}^1L_{n-2+2(N-k)}^3\\
&+\frac{1}{6}L_n^1L_{n-1+N-k}^1L_{n-2+2(N-k)}^3+\frac{2}{9}L_n^1L_{n+N-k}^1L_{n-2+2(N-k)}^3\\
&+\frac{11}{25}L_{n-1}^1L_{n-1+N-k}^1L_{n-1+2(N-k)}^3+\frac{21}{40}L_n^1L_{n-1+N-k}^1L_{n-1+2(N-k)}^3\\
&+\frac{29}{45}L_n^1L_{n+N-k}^1L_{n-1+2(N-k)}^3+\frac{12}{25}L_n^1L_{n+N-k}^1L_{n+2(N-k)}^3\\
&+\frac{6}{25}L_{n-2}^1L_{n-2+N-k}^3L_{n+4(N-k)}^1+\frac{3}{10}L_{n-1}^1L_{n-2+N-k}^3L_{n+4(N-k)}^1\\
&+\frac{3}{8}L_n^1L_{n-2+N-k}^3L_{n+4(N-k)}^1+\frac{23}{40}L_{n-1}^1L_{n-1+N-k}^3L_{n+4(N-k)}^1\\
&+\frac{2}{3}L_n^1L_{n-1+N-k}^3L_{n+4(N-k)}^1+\frac{3}{8}L_n^1L_{n+N-k}^3L_{n+4(N-k)}^1\\
&+\frac{13}{25}L_{n-1}^1L_{n-1+N-k}^3L_{n+1+4(N-k)}^1+\frac{23}{40}L_n^1L_{n-1+N-k}^3L_{n+1+4(N-k)}^1\\
&+\frac{3}{10}L_n^1L_{n+N-k}^3L_{n+1+4(N-k)}^1+\frac{6}{25}L_n^1L_{n+N-k}^3L_{n+2+4(N-k)}^1\\
&+\frac{12}{25}L_{n-2}^3L_{n+3(N-k)}^1L_{n+4(N-k)}^1+\frac{29}{45}L_{n-1}^3L_{n+3(N-k)}^1L_{n+4(N-k)}^1\\
&+\frac{2}{9}L_n^3L_{n+3(N-k)}^1L_{n+4(N-k)}^1+\frac{21}{40}L_{n-1}^3L_{n+1+3(N-k)}^1L_{n+4(N-k)}^1\\
&+\frac{1}{6}L_n^3L_{n+1+3(N-k)}^1L_{n+4(N-k)}^1+\frac{1}{8}L_n^3L_{n+2+3(N-k)}^1L_{n+4(N-k)}^1\\
&+\frac{11}{25}L_{n-1}^3L_{n+1+3(N-k)}^1L_{n+1+4(N-k)}^1+\frac{2}{15}L_n^3L_{n+1+3(N-k)}^1L_{n+1+4(N-k)}^1\\
&+\frac{1}{10}L_n^3L_{n+2+3(N-k)}^1L_{n+1+4(N-k)}^1+\frac{2}{25}L_n^3L_{n+2+3(N-k)}^1L_{n+2+4(N-k)}^1\\
&+\frac{3}{25}L_{n-2}^1L_{n-2+N-k}^2L_{n-1+3(N-k)}^2+\frac{3}{20}L_{n-1}^1L_{n-2+N-k}^2L_{n-1+3(N-k)}^2\\
&+\frac{3}{16}L_n^1L_{n-2+N-k}^2L_{n-1+3(N-k)}^2+\frac{3}{10}L_{n-1}^1L_{n-1+N-k}^2L_{n-1+3(N-k)}^2\\
&+\frac{3}{8}L_n^1L_{n-1+N-k}^2L_{n-1+3(N-k)}^2+\frac{1}{3}L_n^1L_{n+N-k}^2L_{n-1+3(N-k)}^2\\
&+\frac{14}{25}L_{n-1}^1L_{n-1+N-k}^2L_{n+3(N-k)}^2+\frac{53}{80}L_n^1L_{n-1+N-k}^2L_{n+3(N-k)}^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{8}{15}L_n^1L_{n+N-k}^2L_{n+3(N-k)}^2+\frac{8}{25}L_n^1L_{n+N-k}^2L_{n+1+3(N-k)}^2\\
&+\frac{4}{25}L_{n-2}^2L_{n-1+2(N-k)}^1L_{n-1+3(N-k)}^2+\frac{4}{15}L_{n-1}^2L_{n-1+2(N-k)}^1L_{n-1+3(N-k)}^2\\
&+\frac{1}{6}L_n^2L_{n-1+2(N-k)}^1L_{n-1+3(N-k)}^2+\frac{2}{5}L_{n-1}^2L_{n+2(N-k)}^1L_{n-1+3(N-k)}^2\\
&+\frac{1}{4}L_n^2L_{n+2(N-k)}^1L_{n-1+3(N-k)}^2+\frac{1}{6}L_n^2L_{n+1+2(N-k)}^1L_{n-1+3(N-k)}^2\\
&+\frac{17}{25}L_{n-1}^2L_{n+2(N-k)}^1L_{n+3(N-k)}^2+\frac{2}{5}L_n^2L_{n+2(N-k)}^1L_{n+3(N-k)}^2\\
&+\frac{4}{15}L_n^2L_{n+1+2(N-k)}^1L_{n+3(N-k)}^2+\frac{4}{25}L_n^2L_{n+1+2(N-k)}^1L_{n+1+3(N-k)}^2\\
&+\frac{8}{25}L_{n-2}^2L_{n-1+2(N-k)}^2L_{n+4(N-k)}^1+\frac{8}{15}L_{n-1}^2L_{n-1+2(N-k)}^2L_{n+4(N-k)}^1\\
&+\frac{1}{3}L_n^2L_{n-1+2(N-k)}^2L_{n+4(N-k)}^1+\frac{53}{80}L_{n-1}^2L_{n+2(N-k)}^2L_{n+4(N-k)}^1\\
&+\frac{3}{8}L_n^2L_{n+2(N-k)}^2L_{n+4(N-k)}^1+\frac{3}{16}L_n^2L_{n+1+2(N-k)}^2L_{n+4(N-k)}^1\\
&+\frac{14}{25}L_{n-1}^2L_{n+2(N-k)}^2L_{n+1+4(N-k)}^1+\frac{3}{10}L_n^2L_{n+2(N-k)}^2L_{n+1+4(N-k)}^1\\
&+\frac{3}{20}L_n^2L_{n+1+2(N-k)}^2L_{n+1+4(N-k)}^1+\frac{3}{25}L_n^2L_{n+1+2(N-k)}^2L_{n+2+4(N-k)}^1\\
&+\frac{1}{5}L_{n-1}^1L_{n-1+N-k}^1L_{n-1+2(N-k)}^1L_{n-1+3(N-k)}^2+\frac{1}{4}L_n^1L_{n-1+N-k}^1L_{n-1+2(N-k)}^1L_{n-1+3(N-k)}^2\\
&+\frac{1}{3}L_n^1L_{n+N-k}^1L_{n-1+2(N-k)}^1L_{n-1+3(N-k)}^2+\frac{1}{2}L_n^1L_{n+N-k}^1L_{n+2(N-k)}^1L_{n-1+3(N-k)}^2\\
&+\frac{4}{5}L_n^1L_{n+N-k}^1L_{n+2(N-k)}^1L_{n+3(N-k)}^2+\frac{2}{5}L_{n-1}^1L_{n-1+N-k}^1L_{n-1+2(N-k)}^2L_{n+4(N-k)}^1\\
&+\frac{1}{2}L_n^1L_{n-1+N-k}^1L_{n-1+2(N-k)}^2L_{n+4(N-k)}^1+\frac{2}{3}L_n^1L_{n+N-k}^1L_{n-1+2(N-k)}^2L_{n+4(N-k)}^1\\
&+\frac{3}{4}L_n^1L_{n+N-k}^1L_{n+2(N-k)}^2L_{n+4(N-k)}^1+\frac{3}{5}L_n^1L_{n+N-k}^1L_{n+2(N-k)}^2L_{n+1+4(N-k)}^1\\
&+\frac{3}{5}L_{n-1}^1L_{n-1+N-k}^2L_{n+3(N-k)}^1L_{n+4(N-k)}^1+\frac{3}{4}L_n^1L_{n-1+N-k}^2L_{n+3(N-k)}^1L_{n+4(N-k)}^1\\
&+\frac{2}{3}L_n^1L_{n+N-k}^2L_{n+3(N-k)}^1L_{n+4(N-k)}^1+\frac{1}{2}L_n^1L_{n+N-k}^2L_{n+1+3(N-k)}^1L_{n+4(N-k)}^1\\
&+\frac{2}{5}L_n^1L_{n+N-k}^2L_{n+1+3(N-k)}^1L_{n+1+4(N-k)}^1+\frac{4}{5}L_{n-1}^2L_{n+2(N-k)}^1L_{n+3(N-k)}^1L_{n+4(N-k)}^1\\
&+\frac{1}{2}L_n^2L_{n+2(N-k)}^1L_{n+3(N-k)}^1L_{n+4(N-k)}^1+\frac{1}{3}L_n^2L_{n+1+2(N-k)}^1L_{n+3(N-k)}^1L_{n+4(N-k)}^1\\
&+\frac{1}{4}L_n^2L_{n+1+2(N-k)}^1L_{n+1+3(N-k)}^1L_{n+4(N-k)}^1+\frac{1}{5}L_n^2L_{n+1+2(N-k)}^1L_{n+1+3(N-k)}^1L_{n+1+4(N-k)}^1\\
&+L_n^1L_{n+N-k}^1L_{n+2(N-k)}^1L_{n+3(N-k)}^1L_{n+4(N-k)}^1 \qquad (5.100)
\end{aligned}
$$

This formula correctly predicts $L_m^{N,k,5}$ in $N-k\geq 1$ region and reproduce coefficients of hypergeometric functions in $N-k=0$ region if we start from $N\geq 2k$ region and input Schubert numbers. Of course, we can inductively obtain recursive formula for curves of higher degree using the same method, but general structure of the coefficients that appear in recursive formula is still an open problem.

6 Conclusion

In this paper, we proved the fact that correlation functions of hypersurfaces M_N^k in CP^{N-1} $(N \geq k)$ can be written as polynomials of finite number of integers L_m^k up to degree 3. In quintic case, these numbers are 1345, 770, 120. We cannot tell how these results are used in the future, but in proving this, we found the recursion relations that is invariant in $c_1(M_N^k) \geq 2$ case produces "bare" B-model or "bare" coordinates of deformation of complex structure of mirror manifold of M_k^k. This completely agrees with the results of Givental, which saids that in $c_1(M_N^k) \geq 2$ case, sigma models on (M_N^k) can be solved with hypergeometric series without coordinate transformation i.e., (bare deformation parameter is good coordinate of A-model) and that in Calabi-Yau case, we have to translate the bare coordinate by mirror map. In sum, we can say B-model as toric quantum cohomology compatible with toric compactification of moduli space of pure matter theory. And in Calabi-Yau case, we have to introduce mirror map to compensate for the gap between toric compactification of moduli space of pure matter theory and exact moduli space. These conclusion agrees with the argument of [24]. Maybe application of complete intersections in CP^{N-1} can be achieved by changing the input integers L_m^k. We also have to search for the generalization of specialization arguments to the case of weighted projective space. In this case, we would find toric structure of quantum cohomology ring by construction of recursion relations.

Our last step in discussion of Calabi-Yau hypersurfaces in CP^{N-1} is construction of correspondence between correction terms and boundary parts of toric compactifications of moduli space.

Acknowledgment A.C. thanks to A.B. Givental, J. Lewis, C. Peters, S.A.Strømme, G. Tian, Tyurin.

M.J. thanks to T. Eguchi, K. Hori, A. Matsuo, M. Kobayashi, members of Algebraic Geometry Groups in Dept. of Mathematical Science in Tokyo Univ, M. Nagura and Y. Sun for discussions and kind encouragement. A.C. has been partially supported by Science Project Geom. of Alg. Var. n. 0-198-SC1 , and by fundings from M.U.R.S.T. and G.N.S.A.G.A. (C.N.R.) Italy. M.J has been supported by grant of Japanese Ministry of Education.

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